وزارة التعليم العالي والبحث العلمي

*Ministry of Higher Education and Scientific Research*


**BADJI MOKHTAR-ANNABA UNIVERSITY**

**UNIVERSITE BADJI MOKHTAR ANNABA**

جامعة باجي مختار

- عنابة -

**Faculty of Sciences**
**Department of Mathematics**


**Year: 2024/2025**

# THESIS

Presented with a view to obtaining the doctorate degree

## On the Beta-New XLindley Distribution: Simulation and Application in Medicine

**Stream**

**Applied Mathematics**

**Speciality**

**Probability and Statistics**


**By**
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# THÈSE

Présentée en vue de l'obtention du diplôme de Doctorat

## Sur la Distribution Beta-New XLindley: Simulation et Application en Médecine

**Filière**
**Mathématiques Appliquées**

**Spécialité**
**Probabilités et Statistique**


**Par**
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# Contents

# Abstract

This thesis introduces three new distributions: the truncated New-XLindley distribution, the two-parameter beta-exponential distribution, and the New-XLindley beta distribution. These distributions will be useful additions to the current literature on probability theory and data modeling for actuarial and life sciences. We investigate the moment-generating function, order statistics, entropy, and constraint-force reliability of these distributions. The unknown parameters associated with these distributions are estimated using a variety of methods, such as least squares and maximum likelihood. To illustrate the utility of the proposed models, we apply them to a medical data and other data sets. Our goal is to draw in academics by showcasing the versatility and possible applications of these novel distributions.

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**Keywords**: Lindley distribution, new-XLindley distribution, beta distribution, truncated distribution, maximum likelihood method.

# Resumé

Trois nouvelles distributions ont été introduites dans cette thèse, à savoir la distribution bêta de New-XLindley, la distribution bêta-exponentielle à deux paramètres et la distribution tronquée de New-XLindley. L'ajout de ces distributions contribuera à enrichir la littérature existante sur la théorie des probabilités et la modélisation de données pour l'actuariat et les sciences de la vie. Différentes caractéristiques mathématiques de ces distributions sont examinées : les statistiques d'ordre, l'entropie, la fonction génératrice de moments et la fiabilité contrainteforce. Différentes méthodes sont employées afin d'évaluer les paramètres inconnus associés à ces distributions, tels que le maximum de probabilité et les carrés les plus petits. Les modèles proposés sont utilisés pour étudier divers ensembles de données liés à la santé et à d'autres données, dans le but de prouver leur utilité. Nous visons à attirer les chercheurs et à montrer l'adaptabilité et les possibilités d'utilisation de ces nouvelles distributions.

**Keywords**: Distribution de Lindley, Distribution de new-XLindley, Distribution de beta, Distribution tronquée, maximum de vraisemblance.

# الملخص

تُقدّم هذه الأطروحة ثلاثة توزيعات جديدة: توزيع نيو-إكس-ليندلي المُقتطع، وتوزيع بيتا الأسّي ثنائي المعامل، وتوزيع بيتا نيو-إكس-ليندلي. تُشكّل هذه التوزيعات إضافاتٍ قيّمة للأدبيات الحالية حول نظرية الاحتمالات ونمذجة البيانات في العلوم الاكتوارية وعلوم الحياة. ندرس دالة توليد العزوم، وإحصاءات الترتيب، والإنتروبيا، وموثوقية قوى القيد لهذه التوزيعات. تُقدّر المعاملات المجهولة المرتبطة بهذه التوزيعات باستخدام مجموعة متنوعة من الطرق، مثل المربعات الصغرى والاحتمالية القصوى. ولتوضيح فائدة النماذج المقترحة، نُطبّقها على بيانات طبية ومجموعات بيانات أخرى. هدفنا هو جذب الأكاديميين من خلال إبراز تنوع هذه التوزيعات الجديدة وتطبيقاتها المُمكنة.

# Gratitude

This thesis is the culmination of several years of work, reflection, and perseverance. It would not have been possible without the support, guidance, and encouragement of many people I would like to thank.

First, I would like to express my deep gratitude to my thesis director, Pr. **Zeghdoudi Halim**, for his kind supervision, his enlightened advice, and his invaluable patience. His expertise and scientific rigor were essential to the advancement of my research and allowed me to progress throughout this work.

I would also like to thank the jury members for their valuable remarks and suggestions that enriched this thesis. Their critical and constructive perspective was a major asset in the finalization of this work.

A huge thank you to my colleagues and friends from the LaPS laboratory, with whom I shared much more than scientific discussions. Their support, their spirit of camaraderie, and these moments of sharing made these years of research much more enjoyable.

I will not forget my family, who have always supported me with patience and encouragement. Their unwavering confidence in me has been a valuable source of motivation, especially in times of doubt.

Finally, I would like to thank all the people who, directly or indirectly, have contributed to this doctoral adventure. Whether through their advice, moral support, or simply their presence, they have played a fundamental role in the completion of this work.

# Introduction in French

La modélisation et l'analyse des données de survie revêtent une grande importance dans de nombreuses disciplines appliquées, telles que la santé, l'ingénierie et la finance. Des distributions de durée de vie variées ont été employées pour modéliser ces données. Le choix du modèle ou de la distribution probabiliste a un impact significatif sur l'efficacité des méthodes appliquées dans une enquête statistique. Par conséquent, des efforts considérables ont été consacrés à l'élaboration de larges catégories de distributions de probabilités classiques et de méthodes statistiques correspondantes. Cependant, les données empiriques contestent fréquemment les modèles probabilistes établis, engendrant de nombreux défis majeurs non résolus.

Récemment, plusieurs distributions bêta généralisées ont été mises en avant. Les distributions bêta-normale, bêta-Fréchet, bêta-Gumbel, bêta-exponentielle, bêta-généralisée demi-normale, bêta-généralisée Rayleigh, bêta-exponentielle généralisée et bêta-Lindley ont été proposées par Eugene et al. [22], Nadarajah et Gupta [53], Nadarajah et Kotz [54], Nadarajah et Kotz [55], Pescim et al. [58], Cordeiro et al. [18], Barreto et al. [11], ainsi que Merovci et Sharma [48]. L'analyse de Jones et Chris [35] explore cette famille bêta généralisée, mettant en lumière ses propriétés distributionnelles convaincantes et son potentiel pour des applications statistiques intéressantes. Les statistiques d'ordre de ces distributions constituent les fondements de cette discussion.

Dans cette thèse, nous présentons trois nouvelles distributions : la distribution bêta new-XLindley, la distribution bêta-exponentielle à deux paramètres et la distribution tronquée new-XLindley. Pour les deux premières distributions, nous avons employé la famille des distributions générées par bêta ; se référer à Eugene et al. [22], pour introduire une nouvelle généralisation de la distribution XLindley. Dans le domaine des statistiques bayésiennes, la distribution XLindley a été initialement introduite par Nawel et al. [38].

Ces derniers ont analysé les différentes propriétés statistiques de la distribution XLindley. En outre, cette recherche utilise une simulation de Monte Carlo afin d'évaluer et de comparer les performances de divers estimateurs dans l'estimation du paramètre inconnu de la distribution XLindley. Ce modèle a été confronté à de nombreuses distributions existantes telles que XLindley [17], Weibull, gamma exponentielle, Zeghdoudi [49], Akash [59], Lindley [29], Chris-Jerry [58], Shanker [65] et Xgamma [64]. Parmi tous les modèles, il a été déterminé que la nouvelle distribution à un paramètre a produit les résultats les plus performants en matière de modélisation, établis à partir de critères tels que le critère d'information d'Akaike, le critère d'information bayésien et d'autres. Concernant la troisième distribution, la nouvelle distribution tronquée XLindley, celle-ci est introduite à la suite de l'observation de l'applicabilité accrue des distributions tronquées. Au cours de la dernière décennie, la distribution XLindley a suscité l'intérêt des chercheurs, des scientifiques et des praticiens de la fiabilité, incitant de nombreux auteurs à étendre son application à diverses distributions parcimonieuses. Parmi ces extensions figurent la distribution bêta-exponentielle [40], la distribution XLindley de puissance [28], la distribution XLindley de puissance discrète [47], la distribution XLindley modifiée [27] et la distribution XLindley de puissance exponentielle [50]. L'attention généralisée accordée à cette distribution a motivé de nombreux chercheurs et experts en fiabilité à explorer ses extensions sous différentes formes parcimonieuses.

### Motivation et objectifs

La rédaction de ce travail a été conditionnée par plusieurs facteurs :

- Bien qu'elles soient associées à la queue de la distribution, ces distributions s'avèrent aisément applicables.
- La définition explicite des propriétés statistiques constitue un processus simple.
- Les nouvelles distributions offrent plusieurs avantages, notamment trois paramètres susceptibles de modéliser efficacement la science actuarielle, l'analyse de survie et d'autres domaines connexes.
- Ces distributions sont appliquées à deux ensembles de données du monde réel, incluant des valeurs tant petites que grandes, et sont soumises à ajustement et analyse.

- Les fonctions de densité de ces distributions peuvent être exprimées sous une forme linéaire. Le modèle linéaire est généralement préféré à d'autres modèles, comme le modèle quadratique, en raison de sa facilité d'interprétation.
- L'efficacité des distributions tronquées sur les variables aléatoires est restreinte à une certaine plage, et ces situations se présentent fréquemment dans différents domaines.

**Organisation de la thèse**

La thèse est structurée comme suit : une introduction à la thèse actuelle est d'abord donnée. Le contexte historique des chercheurs précédents est également examiné. Des fonctions spéciales, des définitions fondamentales et des concepts sont présentés, ainsi que la méthode T-X pour créer des familles de distributions de probabilité continues avec une famille bêta X, abordées au chapitre 1.

Le chapitre II présente la distribution tronquée et l'article sur la distribution tronquée de la nouvelle XLindley avec ses applications, paru dans le Journal of Computational Analysis and Applications (JoCAAA). Cet article présente les propriétés statistiques, la simulation et les applications du nouveau modèle. Le chapitre III présente une nouvelle distribution de probabilité continue appelée distribution bêta-nouvelle XLindley, qui étend la nouvelle distribution XLindley. Diverses propriétés statistiques de ce nouveau modèle sont explorées, notamment la fonction génératrice de moments, le moment, l'entropie, la fiabilité contrainte-résistance et les statistiques d'ordre. Les paramètres inconnus associés à la distribution bêta-nouvelle XLindley sont estimés à l'aide de plusieurs méthodes. Afin de démontrer l'applicabilité du nouveau modèle, une étude d'application a été menée à partir de deux ensembles de données médicales. Nous présentons un cas particulier correspondant à l'article intitulé « Distribution bêta-exponentielle à deux paramètres : propriétés et applications en démographie et géostandards », paru dans la revue MAS Journal of Applied Sciences. Cet article présente les propriétés statistiques, la simulation et les applications des nouveaux modèles.

# Introduction in English

Modeling and the analysis of survival data are essential in many applied fields, including engineering, finance, and healthcare. To model such data, several lifetime distributions have been used. The effectiveness of the techniques used in a statistical investigation is significantly impacted by the probability model or distribution selection. As such, a great deal of work has gone into creating broad classes of classical probability distributions and associated statistical techniques. However, empirical evidence frequently defies accepted probability models, leaving a great deal of important questions unanswered.

A number of beta-generalized distributions have been presented recently. Eugene et al.[22], Nadarajah and Gupta[53], Nadarajah and Kotz[54], Nadarajah and Kotz[55], Pescim et al. [58], Cordeiro et al. [18], Barreto et al.[11], and Merovci and Sharma[48] proposed the beta-normal, beta-Fréchet, beta-exponential, beta generalized half-normal, beta generalized Rayleigh, beta-generalized exponential, and beta Lindley distributions. This generalized beta family is explored in depth by Jones and Chris[35], who show its intriguing distributional properties and potential for fascinating statistical applications. This discussion is based on the order statistics of these distributions.

In this thesis, we introduce three new distributions: the beta new-XLindley distribution, the two-parameter beta-exponential distribution, and the truncated new-XLindley distribution. In the first two distributions, we used the family of beta-generated distributions; see Eugene et al. [22], to introduce a novel generalization of the new XLindley distribution. In the framework of Bayesian statistics, the new XLindley distribution was first put up by Nawel et al.[38]. The different statistical characteristics of the new XLindley distribution were discussed. Additionally, a Monte Carlo simulation is used in the study to evaluate and contrast the efficacy of different estimators in estimating the new

XLindley distribution's unknown parameter. This model was compared with many current distributions such as XLindley [17], Weibull, gamma exponential, Zeghdoudi [49], Akash [59], Lindley [29], Chris-Jerry [58], Shanker [65], and Xgamma [64]. Based on a variety of criteria, including the Bayesian information criterion and the Akaike information criterion, it is determined that the new one-parameter distribution outperformed the other models. The truncated new-XLindley distribution, which is the third distribution, is presented as a result of the broad applicability of truncated distributions. In the last ten years, scientists, researchers, and reliability practitioners have shown interest in the new XLindley distribution, which has prompted many authors to expand its application to include different parsimonious distributions. Among these extensions are the Beta-Exponential Distribution [40], the Power New XLindley Distribution [28], the Discrete New XLindley Distribution [47], the Modified XLindley Distribution [27], and the Exponentiated New XLindley Distribution [50]. This distribution has received a lot of attention, which has motivated many scholars and reliability specialists to look into its expansions into other parsimonious forms.

**Motivation and Objectives**

Several factors influenced the creation of this work:

- These distributions turn out to be readily applicable even though they are confided to the tail of the distribution.
- Explicitly defining the statistical properties is a simple procedure.
- Three parameters that can effectively model actuarial science, survival analysis, and other related fields are just one of the novel distributions' many advantages.
- Fitting and analysis are performed on two real-world data sets that contain both small and large values, using these distributions.
- The density functions of these distributions can be expressed in a linear form; due to its ease of interpretation, the linear model is frequently chosen over other models, like the quadratic model.
- There is a limited range in which truncated distributions on random variables are effective, and these kinds of situations are common in many different fields.

### Organization of the Thesis

The thesis is structured as follows: An introduction to the current thesis is given first. The historical context of the earlier researchers is also examined. Some special functions, fundamental definitions, and concepts are presented, and we present the T-X method for creating families of continuous probability distributions with some beta X-family, which are covered in Chapter I.

In Chapter II, the truncated distribution was introduced, and we present the article on the truncated new-XLindley distribution with applications that appeared in the Journal of Computational Analysis and Applications (JoCAAA). This paper presents the statistical properties, simulation, and applications of the new model. In Chapter III, we introduce a novel continuous probability distribution called the beta-new XLindley distribution, which extends the new XLindley distribution. Various statistical properties of this new model are explored, including the moment-generating function, the moment, entropy, stress-strength reliability, and order statistics. The unknown parameters associated with the beta-new XLindley distribution are estimated using several methods. To demonstrate the applicability of the new model, an application study using two medical data sets was conducted. And we show a special case that corresponds to the article entitled "Two-Parameter Beta-Exponential Distribution: Properties and Applications in Demography and Geostandards, which appeared in the MAS Journal of Applied Sciences. This paper presents the statistical properties, simulation, and applications of the new models.

# Chapter 1

# Basic Concepts and Models

As part of this section, we discuss a few distributions, special functions, and estimation methods that we used throughout the thesis, such as maximum likelihood estimation (MLE), least squares and weighted least squares estimators (LSE/WLSE), and others. And we present the T-X method for creating families of continuous probability distributions with some beta X-family.

# 1.1 Overview On Some Distributions

## 1.1.1 Beta distribution

The beta distribution is a family of continuous probability distributions with two positive shape parameters, denoted by $\alpha$ and $\beta$, and a range of values on the interval $[0, 1]$. These two parameters appear as exponents of the random variable and govern the form of the distribution. A "beta distribution" is the fundamental distribution of the first kind, while a "prime beta distribution" is the distribution of the second kind. Most frequently, this distribution is employed to simulate the uncertainty surrounding the likelihood that a random experiment will succeed. A three-point method called the "beta distribution" is employed in project management to recognize the degree of uncertainty in the project time estimate. It computes the confidence levels for the expected completion time using robust quantitative tools and basic statistics.

**Definition 1.1.1** *For $0 \leq x \leq 1$ or $0 < x < 1$, and shape parameters $\alpha, \beta > 0$, the probability density function (pdf) of the beta distribution is a power function of the variable $x$ and of its reflection $(1 - x)$ in the following way:*

$$f(x, \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}.$$

*where* $B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, *with* $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$ *is Gamma function.*

*The cumulative distribution function (cdf) is given by:*

$$F(x, \alpha, \beta) = \frac{B(x, \alpha, \beta)}{B(\alpha, \beta)} = I_x(\alpha, \beta).$$

*where $B(x, \alpha, \beta)$ is the incomplete beta function and $I_x(\alpha, \beta)$ is the regularized incomplete beta function.*

**Properties**

**1** If $X \sim Beta(\alpha, \beta)$ then the mean and variance of $X$ are given respectively by:

$$\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta},$$

$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

**2** If $X \sim Beta\,(\alpha, \beta)$ then the skewness and kurtosis of $X$ are given respectively by:

$$\sqrt{\beta_1} = \frac{2\,(\beta - \alpha)\sqrt{\alpha + \beta + 1}}{(\alpha + \beta + 2)\sqrt{\alpha\beta}}.$$

$$\beta_2 = \frac{6\left[(\beta - \alpha)^2 (\alpha + \beta + 1) - \alpha\beta\,(\alpha + \beta + 2)\right]}{\alpha\beta\,(\alpha + \beta + 2)\,(\alpha + \beta + 3)}.$$

### 1.1.2 Gamma distribution

Because it has so many practical uses, the gamma distribution is one of the most widely used and well-liked continuous time distributions. It covers a wide range of life events, such as the likelihood of rain, the dependability of mechanical tools and equipment, or any uses that yield only favorable outcomes. The skewed shape of the Gamma distribution can be explained by the unfortunate fact that these applications are frequently unbalanced. The gamma distribution law is predicated on the corresponding function $\Gamma$.

This function, in the form of an elementary function, is an integral that admits no primitive. It is written:

$$\Gamma\,(n) = \int_0^{\infty} x^{n-1} \exp\,(-x)\,dx \quad \text{with } n > 0.$$

And we have also

$$\Gamma\,(n+1) = n\Gamma\,(n) = n!.$$

**Definition 1.1.2** *A continuous random variable $X$ follows a Gamma distribution with parameters $\alpha$ and $\beta$ (strictly positive), its probability density function is written:*

$$f(x) = \frac{\beta^{\alpha}}{\Gamma\,(\alpha)} x^{\alpha-1} \exp\,(-\beta x) \qquad \textit{for} \;\; x > 0.$$

*The cumulative distribution function (cdf) and survival function are given by:*

$$F(x) = \frac{\gamma\,(\alpha, \beta x)}{\Gamma\,(\alpha)} \qquad \textit{and} \qquad S(x) = 1 - \frac{\gamma\,(\alpha, \beta x)}{\Gamma\,(\alpha)}.$$

*where $\gamma\,(\alpha, \beta x)$ is the lower incomplete gamma function.*

**Remark 1.1.1** *If $\alpha = 1$, then Gamma distribution is the exponential distribution with parameter $\beta$.*

**Properties**

1. If $X \sim G(\alpha, \beta)$ then the mean and variance of $X$ are given respectively by:

$$\mathbb{E}(X) = \frac{\alpha}{\beta},$$
$$\mathrm{Var}(X) = \frac{\alpha}{\beta^2}.$$

2. If $X \sim G(\alpha, \beta)$ then the skewness and kurtosis of $X$ are given respectively by:

$$\sqrt{\beta_1} = \frac{2}{\sqrt{\alpha}},$$
$$\beta_2 = \frac{6}{\alpha}.$$

### 1.1.3 Lindley Distribution

Researchers were drawn to the Lindley distribution of a single parameter because of its potential for modeling real-world data, and multiple articles have noted how well this distribution worked. In 1958, Lindley presented this distribution as a combination of $Exp(\theta)$ and $Gamma(2;\theta)$

**Definition 1.1.3** *A continuous random variable $X$ follows a Lindley distribution with parameters $\theta$ ($\theta$ is strictly positive), its probability density function is written:*

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x) \qquad \textit{with} \qquad x > 0.$$

*The corresponding cumulative distribution function (cdf) is:*

$$F(x) = 1 - \frac{\theta+1+\theta x}{\theta+1}\exp(-\theta x) \qquad \textit{with} \qquad x > 0.$$

*The corresponding survival function is given by:*

$$S(x) = \frac{\theta+1+\theta x}{\theta+1}\exp(-\theta x) \qquad \textit{with} \qquad x > 0.$$

**Properties**

1. The $k$th moment about the origin of the Lindley distribution is given by:

$$\mu_k' = \mathbb{E}(X^k) = \frac{k!\,(\theta+k+1)}{\theta^k(\theta+1)}.$$

2. The central moments of the Lindley distribution are:

$$\mu_k = \mathbb{E}((X-\mu)^k) = \sum_{i=0}^{k} C_i^k \mu_k^{'} (-\mu)^{k-i}.$$

3. The coefficient of variation ($\gamma$), skewness ($\sqrt{\beta_1}$) and the kurtosis ($\beta_2$) are:

$$\gamma = \frac{\sqrt{\theta^2+4\theta+2}}{\theta+2},$$
$$\sqrt{\beta_1} = \frac{2\left(\theta^3+6\theta^2+6\theta+2\right)}{\left(\theta^2+4\theta+2\right)^{\frac{3}{2}}}, \text{ and}$$
$$\beta_2 = \frac{3\left(3\theta^4+24\theta^3+44\theta^2+32\theta+8\right)}{\left(\theta^2+4\theta+2\right)^2}.$$

### 1.1.4 XLindley Distribution

The exponential distribution and the Lindley distribution with parameter $\theta$ are two known distributions that are mixed to create the XLindley distribution, which is a probability distribution. It is a probability distribution that is continuous and defined on the positive real numbers. In reliability modeling and survival analysis, the XLindley distribution is frequently utilized.

**Definition 1.1.4** *Let $X$ be a random variable following mixture distribution, it's density function (pdf) $f(x)$ given as:*

$$f(x) = \sum_{i=1}^{k} p_i f_i(x).$$

*where*

- $f_i(x)$ *probability density function for each* $i$.
- $p_i$ *denote mixing proportions that are no-negative and* $\sum_{i=1}^{k} p_i = 1$.

*So we get the XLindley distribution when we take* $f_1(x) \sim Exp(\theta)$ *and* $f_2(x) \sim LD(\theta)$ *two independents random variables with* $p_1 = \frac{\theta}{\theta+1}$ *and* $p_2 = 1 - \frac{\theta}{\theta+1}$ *respectively.*

*Now the (pdf) of XLindley distribution is given by:*

$$f(x) = \frac{\theta^2}{(\theta+1)^2}(\theta+x+2)\exp(-\theta x) \qquad \text{with} \qquad \theta, x > 0.$$

*The cumulative distribution function (cdf) of the XLD:*

$$F(x) = 1 - \left(1 + \frac{\theta x}{(\theta+1)^2}\right)\exp(-\theta x) \qquad \textit{with} \qquad \theta, x > 0.$$

*The survival function and failure rate (hazard rate) function are given as, respectively:*

$$S(x) = \left(1 + \frac{\theta x}{(\theta+1)^2}\right)\exp(-\theta x) \qquad \textit{with} \qquad \theta, x > 0,$$
$$H(x) = \frac{\theta^2(\theta + x + 2)}{(\theta+1)^2 + x\theta} \qquad \textit{with} \qquad \theta, x > 0.$$

**Properties**

1. The $k$th moment of the XLindey distribution is given by:

$$\mu_k^{'} = \frac{k!\left(\theta^2 + 2\theta + k + 1\right)}{(\theta+1)^2\theta^k}.$$

2. The mean, variance, coefficients of variation, skewness and kurtosis for $X$ are:

$$\mathbb{E}(X) = \frac{\left(\theta^2 + 2\theta + 2\right)}{(\theta+1)^2\theta},$$
$$\mathrm{Var}(X) = \frac{(\theta+1)^4 + 4\theta^2 + 6\theta + 1}{(\theta+1)^4\theta^2},$$
$$\gamma = \frac{\sqrt{\mathrm{Var}(X)}}{\mathbb{E}(X)} = \frac{\sqrt{(\theta+1)^4 + 4\theta^2 + 6\theta + 1}}{(\theta+1)^2 + 1},$$
$$\sqrt{\beta_1} = \frac{\mathbb{E}(X^3)}{(\mathrm{Var}(X))^{\frac{3}{2}}} = \frac{6(\theta+1)^4\left(\theta^2 + 2\theta + 4\right)}{\left((\theta+1)^4 + 4\theta^2 + 6\theta + 1\right)^{\frac{3}{2}}}, \text{ and}$$
$$\beta_2 = \frac{\mathbb{E}(X^4)}{(\mathrm{Var}(X))^2} = \frac{24\left(\theta^2 + 2\theta + 5\right)(\theta+1)^6}{\left((\theta+1)^4 + 4\theta^2 + 6\theta + 1\right)^2}.$$

### 1.1.5 New-XLindley Distribution

The Lindley and exponential distributions' advantages are combined in the one-parameter New-XLindley distribution. It can be applied to a number of fields, such as actuarial science, biology, engineering, astronomy, and medicine. Conversely, the new distribution features a declining average residual life function and an elevated risk rate.

**Definition 1.1.5** *A new statistical family called the new one-parameter polynomial exponential distribution (NPED) was recently introduced by Beghriche et al.[12] . With the* *probability density function (pdf) :*

$$f(x)=\frac{\sum\limits_{k=0}^{n} x^k a_{k,\theta}}{\sum\limits_{k=0}^{n} \frac{k!}{\theta^{k+1}} a_{k,\theta}} \exp(-\theta x) \qquad with \qquad \theta, x>0. \tag{1.1}$$

*The New-XLindley distribution is obtained as a special case of 1.1, when* $a_{0,\theta}=1$, $a_{1,\theta}=\theta$ *and* $n=1$ *, and the corresponding (pdf) is given by:*

$$f(x)=\frac{\theta}{2}(\theta x+1)\exp(-\theta x) \qquad with \qquad \theta, x>0.$$

*and it can be obtained by mixture of* $f_1(x)\sim Exp(\theta)$ *and* $f_2(x)\sim G(2,\theta)$ *with* $p_1=p_2=\frac{1}{2}$.

*The cumulative distribution function (cdf) of the NXLD is defied as follows:*

$$F(x)=1-\left(\frac{\theta x}{2}+1\right)\exp(-\theta x) \qquad with \qquad \theta, x>0.$$

*The survival function* $S(x)$ *and hazard rate function* $H(x)$ *for the NXLD are, respectively, defied as follows:*

$$S(x)=\left(\frac{\theta x}{2}+1\right)\exp(-\theta x) \qquad with \qquad \theta, x>0,$$
$$H(x)=\frac{\theta+\theta^2 x}{\theta x+2} \qquad with \qquad \theta, x>0.$$

**Property**

1. The $k$th moment of the NXLD is defied as follows:

$$\mu_k^{'}=\frac{1}{2\theta^k}\left[\Gamma(k+1)+\Gamma(k+2)\right].$$

The mean, variance, coeffiients of variation, skewness, and kurtosis for $X$ are, respectively defied as follows:

$$\mathbb{E}(X)=\frac{3}{2\theta}, \qquad \mathrm{Var}(X)=\frac{7}{4\theta^2},$$
$$\gamma=\frac{\sqrt{\frac{7}{4\theta^2}}}{\frac{3}{2\theta}}=\frac{\sqrt{7}}{3},$$
$$\sqrt{\beta_1}=\frac{\frac{15}{\theta^3}}{\left(\frac{7}{4\theta^2}\right)^{\frac{3}{2}}}=\frac{120}{49}\sqrt{7}, \text{ and}$$
$$\beta_2=\frac{\frac{72}{\theta^4}}{\left(\frac{7}{4\theta^2}\right)^2}=\frac{1152}{49}.$$

## 1.2 Some Special Functions

We defined a few special functions in this subsection that are necessary for the remainder of the thesis, such as the well-known incomplete gamma function, which is defined by:

$$\Gamma(\theta, x) = \int_0^x t^{\theta-1} \exp(-t)\, dt \qquad \text{where} \qquad \theta > 0.$$

The beta function ( $\Gamma(\cdot)$ is the gamma function) given by:

$$B(a,b) = \int_0^\infty t^{a-1}(1-t)^{b-1}\, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The cumulative distribution function (cdf) of the beta distribution with parameters $a$ and $b$, or the incomplete beta function ratio, is defined by:

$$I_y(a,b) = \frac{1}{B(a,b)} \int_0^y t^{a-1}(1-t)^{b-1}\, dt.$$

The function of confluent hypergeometric defined by:

$${}_1F_1(a,b,z) = \sum_{i=0}^{\infty} \frac{(a)_i z^i}{i!\,(b)_i}.$$

Where $(a)_i$ is the ascending factorial defined by (with the convention that $(a)_0 = 1$)

$$(a)_i = a(a+1)(a+2)\ldots(a+i-1).$$

The function of Gaussian hypergeometric defined by:

$${}_2F_1(a,b,c,z) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i z^i}{i!\,(c)_i}.$$

The Lauricella function of type A [Exton (1978)[23]; Aarts (2000)[1]] defined by:

$$F_A^{(n)}(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}}.$$

The generalized Kampé de Fériet function [Exton (1978); Mathai (1993)[46]; Aarts (2000)[1]; Chaudhry and Zubair (2002)[14]] defined by:

$$F_{C:D}^{A:B}((a):(b_1),\ldots,(b_n);(c):(d_1),\ldots,(d_n);x_1,\ldots,x_n) = \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} \times \frac{((a))_{m_1+\ldots+m_n}((b_1))_{m_1} \ldots ((b_n))_{m_n}}{((c))_{m_1+\ldots+m_n}((d_1))_{m_1} \ldots ((d_n))_{m_n}}.$$

Where $a = (a_1, a_2, \ldots, a_A)$, $b_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,B})$, $c = (c_1, c_2, \ldots, c_A)$, $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,B})$ for $i = 1, 2, .., n$, and $((f))_k = ((f_1, f_2, \ldots, f_p))_k = (f_1)_k (f_2)_k \ldots (f_p)_k$ denotes the product of ascending factorials.

## 1.3 Estimation Methods

We outline the various techniques we employed to estimate the parameters in this section.

### 1.3.1 Maximum Likelihood Estimation (MLE)

The maximum likelihood method, which was created by statistician Ronald Fisher between 1912 and 1922 [63], is a statistical estimation technique that is frequently used to infer the parameters of the probability distribution of a given sample.

Let $X$ be a random variable of any law, depending on a parameter $\theta$ that we want to estimate, and let $x_1, x_2, ..., x_n$ be a realization of the theoretical sample $X_1, X_2, ..., X_n$. from the random variable $X$ which admits $f(x)$ as probability density. We call the likelihood function noted $L(x_1, x_2, ..., x_n, \theta)$ the random variable (or a function of the variable $\theta$) defined as follows:

$$L(x_1, x_2, ..., x_n, \theta) = \prod_{i=1}^{n} f(x_i).$$

The search for the MLE can be done, under certain conditions, in a direct manner, and this by the search for the maximum extremum of $L$, that is to say that, when the function $L$ is twice differentiable with respect to $\theta$, it suffices to verify the following points:

$$\begin{cases} \dfrac{dL(x_1, x_2, ..., x_n, \theta)}{d\theta} = 0, & \text{the solution to this equation provided } \hat{\theta}. \\ \dfrac{d^2L(x_1, x_2, ..., x_n, \theta)}{d\theta^2} < 0, & \text{to ensure that } \hat{\theta} \text{ is a maximum extremum.} \end{cases} \tag{1.2}$$

By definition, the likelihood is calculated from a product of $n$ elements. However, we prefer to replace the problem defined in 1.2 with a less complex equivalent problem. Since the natural logarithm function (log) is strictly increasing and $L(x_1, x_2, ..., x_n, \theta)$ and $\log L(x_1, x_2, ..., x_n, \theta)$ reach their maximums for the same value of $\theta$, it would often be easier to solve the following equivalent system of equations:

$$\begin{cases} \dfrac{d \log L(x_1, x_2, ..., x_n, \theta)}{d\theta} = 0, \\ \dfrac{d^2 \log L(x_1, x_2, ..., x_n, \theta)}{d\theta^2} < 0. \end{cases}$$

It should be noted that, in the case where $\theta$ is a parameter of dimension $k$ (*i.e.* $\theta = (\theta_1, ..., \theta_n)$), we solve a system of $k$ equations, the latter of which are obtained by differentiating $L(x_1, x_2, ..., x_n, \theta)$ with respect to each of the components of $\theta$.

### 1.3.2 Least Squares and Weighted Least Squares Estimators (LSE/WLSE)

This type of approximation known as the least squares approach is attributed to Gauss. It is usually used on the parameter estimate of the linear model. The beta distribution parameters were estimated by Swain, Venkatraman, and Wilson (1988)\[67] using weighted least squares and least squares estimates. Additionally, refer to Gupta and Kundu (2001)[30], Kundu and Raqab (2005)[42], and Alkasabeh and Raqab (2009)[5].

Let $X_1, X_2, ..., X_n$ be order statistics from a random sample of size $n$ from a distribution function $G(.)$ and suppose $X_{(i)}$ denotes the ordered sample. The proposed method uses the distribution of $G(X_{(i)})$, $i = 1, 2, .., n$. For a sample of size $n$ we have:

$$\mathbb{E}\left(G(X_{(i)})\right) = \frac{i}{n+1}, \qquad V\left(G(X_{(i)})\right) = \frac{i\,(n-i+1)}{(n+1)^2\,(n+2)}, \text{ and}$$

$$Cov\left(G(X_{(i)}), G(X_{(j)})\right) = \frac{i\,(n-j+1)}{(n+1)^2\,(n+2)} \quad \text{for } i < j.$$

The least squares estimates can be obtained by minimizing:

$$\sum_{i=1}^{n}\left(G(X_{(i)}) - \frac{i}{n+1}\right)^2.$$

with respect to the unknown parameters.

The weighted least squares estimates of the unknown parameters can be obtained by minimizing:

$$\sum_{i=1}^{n} W_i\left(G(X_{(i)}) - \frac{i}{n+1}\right)^2.$$

Where

$$W_i = \frac{1}{V\left(G(X_{(i)})\right)} = \frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}.$$

with respect to the unknown parameters.

### 1.3.3 Methods of Anderson-Darling and (Right/left)-tail Anderson-Darling (ADE/RTADE)

The Anderson-Darling test was developed in 1952 by Anderson and Darling [7] to detect sample distributions that are not normal. In this sense, the test acted as a substitute for the statistical tests that were already in place. In particular, the AD test converges very quickly towards the asymptote, according to Anderson and Darling (1954)[8], Pettitt (1976) [25], and Stephens (2013) [66].

This function is minimized to yield the parameters' Anderson-Darling estimators:

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left(G(x_{(i)})\right) + \log\left(S(x_{(i)})\right)\right).$$

The Right-tail Anderson-Darling estimates of parameters are obtained by minimizing this function:

$$RAD = \frac{n}{2} - 2\sum_{i=1}^{n} G(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(S(x_{(n-i+1)})\right).$$

The left tailed Anderson–Darling of parameters are obtained by minimizing this function:

$$LAD = -\frac{3}{2}n + 2\sum_{i=1}^{n} G(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(S(x_{(n-i+1)})\right).$$

### 1.3.4 The Cramér-von Mises estimation (CVME)

Our choice of Cramér-von-Mises type minimum distance estimators is empirically supported by Macdonald (1971)[45], who shows that the estimator's bias is less than that of the other minimum distance estimators. Therefore, by minimizing this function, the parameters' Cramér-von-Mises estimate is obtained:

$$C = \frac{1}{12n}\sum_{i=1}^{n}\left(G(x_{(i)}) - \frac{2i-1}{2i}\right)^2.$$

### 1.3.5 Method of Maximum Product of Spacings (MPSE)

Cheng & Amin (1979[15], 1983[16]) introduced the maximum product of spacings (MPS) approach as an alternative to maximum linear equation (MLE) for estimating parameters for continuous univariate distributions. Ranneby (1984)[63] independently developed a similar technique as an estimate for the **Kullback-Leibler** measure of information.

The maximum product of the spacings estimation approach is used to estimate the parameters, necessitating the optimization of the subsequent equation:

$$T = \left( \prod_{i=1}^{n+1} D_i \right)^{\frac{1}{n+1}}.$$

or, equivalently, by maximizing the function

$$\log(T) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i).$$

where $D_i = G(x_{(i)}) - G(x_{(i-1)})$.

### 1.3.6 The minimum spacing absolute distance estimation (MSADE)

The minimum spacing absolute distance estimation approach is used to estimate the parameters, and it involves maximizing the following equation:

$$W = \sum_{i=1}^{n+1} \left| M_i - \frac{1}{n+1} \right|.$$

### 1.3.7 The minimum spacing absolute-log distance estimation (MSALDE)

The minimum spacing absolute-log distance estimation approach is used to estimate the parameters, and it involves maximizing the following equation:

$$W = \sum_{i=1}^{n+1} \left| \log(M_i) - \log\left( \frac{1}{n+1} \right) \right|.$$

## 1.4 T-X Method For Generating Families of Continuous Probability Distributions

Statistical distributions are useful tools for accurately describing and forecasting real-world events. Even though many distributions have been developed, there is always room for distributions that are more adaptable or that address specific real-world circumstances. This has encouraged researchers to find and develop new, flexible distributions. As a result, many new distributions have been developed and studied.

The Pearson system of continuous distributions was created by Pearson in [57]. It is made up of probability density functions $(p.d.f)$ $f(x)$ that satisfy a differential equation of the following form:

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{a+x}{b_0+b_1x+b_2x^2}. \tag{1.3}$$

where the parameters are $a$, $b_0$, $b_1$, and $b_2$. The shape of the function $f(x)$ is determined by the parameters. The different distribution forms were divided into multiple categories by Pearson. The numerous variations correspond to the different approaches to solving (1.3). The roots of the equation, $b_0 + b_1x + b_2x^2 = 0$, indicate the approach to take in solving (1.3). An illustration would be if $b_1 = b_2 = 0$, which produced the normal distribution and is not type-specific.

Burr [13] introduced a system of continuous distributions that can take on a multitude of forms. The distribution system satisfies the differential equation.

$$dF = F(1-F)g(x)dx. \tag{1.4}$$

where $0 \leq F \leq 1$ over $x$ and $g(x)$ is a non-negative function. Equation (1.4) has **12** solutions according to Burr [13], which correspond to $g(x)$ alternatives.

The following general type of normalizing transformation technique for generating distributions was proposed by Johnson [34]:

$$Z = \gamma + \delta f\left(\frac{x-\xi}{\lambda}\right).$$

$Z$ is a standardized normal random variable, $\gamma$ and $\delta$ are shape parameters, $\lambda$ is a scale parameter, $\xi$ is a location parameter, and $f(.)$ is the transformation function. Without sacrificing generality, Johnson [34] accepted the positive values of $\delta$ and $\lambda$. He provided three transformation functions and defined the lognormal family, the bounded system of distributions, and the unbounded system of distributions. These families of distributions include the exponential, gamma, beta, normal, log-normal, and many other commonly used distributions.

Ramberg and Schmeiser [61, 62] and Ramberg et al. [60] generalized the lambda distribution first proposed by Tukey [70] to create what are known as the generalized lambda distributions (GLD). This set of distributions is defined by the percentile function:

$$Q(y) = Q(y; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2} \quad \text{where} \quad 0 \leq y \leq 1.$$

The corresponding $(p.d.f)$ is given by:

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + (1-y)^{\lambda_4 - 1}} \qquad \text{with} \;\; x = Q(y).$$

where the location and scale parameters are indicated by $\lambda_1$ and $\lambda_2$, respectively, and the skewness and kurtosis are determined by $\lambda_3$ and $\lambda_4$.

For a $(p.d.f)$ to exist, it is necessary that $\lambda_3 y^{\lambda_3 - 1} + (1-y)^{\lambda_4 - 1}$ have the same sign for any y in $[0, 1]$ and $\lambda_2$ take the same sign. Freimer et al. [26] conducted a comparison and contrast of Pearson's system and the GLD. They argued that Pearson's family does not contain a logistic distribution, even though GLD does not cover all skewness and kurtosis values. An extended GLD that combines the GLD with the generalized beta distribution -defined as follows- was presented by Karian and Dudewicz [36].

$$f(x) = \begin{cases} \dfrac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{B(\beta_3+1, \beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}} & \text{where} \quad \beta_1 \leq x \leq \beta_1 + \beta_2, \\ 0 & \text{otherwise.} \end{cases}$$

With $B(.,.)$ is the complete beta function.

The skew normal family of distributions was first presented by Azzalini [10]. Let's say that $X$ and $Y$ are independent random variables, each having a symmetric $(p.d.f)$ about zero. For any $\lambda$,

$$P(X - \lambda Y < 0) = \int_{\mathbb{R}} f_Y(y) F_X(\lambda y) dy = \frac{1}{2}.$$

It follows that a probability density function is $2f_Y(y)F_X(\lambda y)$. In the skew-normal family of distributions, if $X$ and $Y$ are both standard normal, $N(0,1)$, then the $(p.d.f)$ represents:

$$2\varphi(x)\Phi(\lambda x). \tag{1.5}$$

where $\varphi(x)$ and $\Phi(\lambda x)$ are $N(0,1)$ $(p.d.f)$ and $(c.d.f)$ respectively.

The distribution in (1.5) is described by a single parameter, $\lambda$. Location and scale parameters can be added to the distribution in (1.5) by using the translation $Y = \mu + \sigma X$. for continuous distribution systems, like the skew normal distribution.

Eugene et al.[22] used the beta distribution as a generator to produce the so-called family of beta-generated distributions. The definition of the cumulative distribution function $(c.d.f.)$ of a random variable $X$ produced by a beta distribution is as follows:

$$G(x) = \int_0^{F(x)} b(t)dt. \tag{1.6}$$

where $b(t)$ is the $(p.d.f)$ of the beta random variable and $F(x)$ is the $(c.d.f)$ of any random variable. So we can write $G(x)$ like this:

$$G(x) = \frac{1}{\beta(a,b)} \int_0^{F(x)} t^{a-1}(1-t)^{b-1}dt \quad \text{where} \quad 0 < a, b < +\infty. \tag{1.7}$$

According to Eugene et al. [22] and Jones [35] , this family of distributions is a generalization of the distributions of order statistics for the random variable $X$ with $(c.d.f)$ $F(x)$.

The beta-generated family of distributions was recently expanded by Jones [35] and Cordeiro and de Castro [19] by substituting the Kumaraswamy distribution, $b(t) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$, $x \in (0,1)$, Kumaraswamy [41] , for the beta distribution in (1.6). The Kumaraswamy generalized distributions (KW-G) $(p.d.f)$ are provided by:

$$g(x) = \alpha\beta f(x)F^{\alpha-1}(x)\left(1 - F^{\alpha}(x)\right)^{\beta-1}.$$

A technique for creating skewed distributions using inverse probability integral transformations was presented by Ferreira and Steel [24] .

Ferreira and Steel [24] state that if a distribution $G$'s $(p.d.f)$ has the following form, it is considered a skewed version of the symmetric distribution $F$ produced by the skewing mechanism $P$:

$$g(y|F,P) = f(y)p(F(y)). \tag{1.8}$$

Observe that the weighted function of $f(.)$ with weight $p(F(.))$ is represented by the $(p.d.f)$ (1.8). One particular example of this family is the skewed normal family in (1.5). By easing the requirement that $F(.)$ be symmetric, the beta-generated family (1.7) can be considered a particular case of (1.8).

### 1.4.1 T-X Method

The distributions in the beta-generated family are generated by using generators that have support values ranging from 0 to 1. The beta random variable and any other random variable's $(c.d.f)$ $F(x)$ both lie between 0 and 1. The limitation of using a generator whose support is between 0 and 1 raises some interesting questions, one of which is *"Can we use other distributions with different support as the generator to derive different classes of distributions?"* This section will provide an answer to the question as well as a new technique for creating families of distributions using any $(p.d.f)$ as a generator.

The $T-X(W)$ family was defined and the generalized beta family was extended by Alzaatreh et al. [6]. The distribution $T-X(W)$'s cumulative distribution function is

$$G(x) = \int_{a}^{W(F(x))} r(t)dt. \tag{1.9}$$

where $r(t)$ the probability density of a random variable $T \in [a,b]$, for $0 < a, b < +\infty$. Consider a function $W(F(x))$ that is a function of the $(c.d.f)$ $F(x)$ of any random variable $X$, satisfying the following requirements:

$$\left.\begin{array}{c} W(F(x)) \in [a,b] \\ W(F(x)) \text{ is both monotonically non-decreasing and differentiable.} \\ \text{if } x \to -\infty : W(F(x)) \to a \text{ and if } x \to +\infty : W(F(x)) \to b \end{array}\right\} \tag{1.10}$$

The following definition presents a procedure for creating new families of distribution.

**Definition 1.4.1** *Let $X$ be a random variable with $(p.d.f)$ $f(x)$ and $(c.d.f)$ $F(x)$. Let $T$ be a continuous random variable with $(p.d.f)$ $r(t)$ defined on $[a,b]$. The $(c.d.f)$ of a new* *family of distributions is defined as*

$$G(x) = \int_a^{W(F(x))} r(t)dt. \tag{1.11}$$

*$W(F(x))$ meets the requirements in (4) in this instance. Since $R(t)$ is the $(c.d.f)$ of the random variable $T$, the $(c.d.f)$ $G(x)$ in (1.11) can be expressed as $G(x) = R\{W(F(x))\}$. For (5), the corresponding $(p.d.f)$ is:*

$$g(x) = \left\{ \frac{d}{dx} W(F(x)) \right\} r\left(W(F(x))\right). \tag{1.12}$$

*Notice that:*

- *A composite function of $(R.W.F)(x)$ is represented by the $(c.d.f)$in (1.11).*
- *Using the function $W(F(x))$, which serves as a "transformer" the $(p.d.f)$ $r(t)$ in (1.11) is "transformed" into a new $(c.d.f)$ $G(x)$. For this reason, we will refer to the distribution $g(x)$ in equation (1.12) as having been "Transformed-Transformer" or "$T - X$" transformed from random variable $T$ through the transformer random variable $X$.*
- *If $X$ is a discrete random variable, then $G(x)$ is the $(c.d.f)$ of a family of discrete distributions.*

A different $W(F(x))$ will result in a new distribution family. The support of the random variable $T$ is necessary for the definition of $W(F(x))$. Here are a few instances of $W(.)$.

1. In cases when $T$'s support is bounded, we presume that it is $[0, 1]$ without sacrificing generality. Uniform $(0, 1)$, beta, Kumaraswamy, and other generalized beta distributions are examples of distributions for such $T$. $F(x)$ or $F^{\alpha}(x)$ can be used to define $W(F(x))$. This is the family of beta-generated distributions that have been thoroughly examined in the last ten years.

2. If $[a, \infty)$ is the support of $T$, then $a \geq 0$: We can suppose $a = 0$ without losing generality. $-log(1-F(x))$, $F(x)/(1-F(x))$, $-log(1-F^{\alpha}(x))$, and $F^{\alpha}(x)/(1-F^{\alpha}(x))$ are the definitions of $W(F(x))$, when $\alpha > 0$.

3. If $(-\infty, \infty)$ is the support of $T$, $W(F(x))$ can be defined as $log[-log(1 - F(x))]$, $log[F(x)/(1 - F(x))]$, $log[-log(1 - F^{\alpha}(x))]$, and $log[F^{\alpha}(x)/(1 - F^{\alpha}(x))]$.

Using the first example's $W(F(x)) = F(x)$, the $G(x)$ is a $(c.d.f)$ of the new family of distributions, which is given by:

$$G(x) = \int_a^{F(x)} r(t)dt = R\left(F(x)\right). \tag{1.13}$$

where $R(t)$ is the $(c.d.f)$ of the random variable $T$. The corresponding $(p.d.f)$ associated with (1.13) is:

$$g(x) = \left\{\frac{d}{dx}W(F(x))\right\} r\left(W(F(x))\right). \tag{1.14}$$

where $F(x)$ is the $(c.d.f)$ for the random variable $X$.

Table1 provides the appropriate families of distributions produced by the other $W(.)$ functions stated in examples 2 and 3.

Table 1.1: Probability density functions of a few T-X families derived from various W(.) functions

| Support of $T$ | $W(F(x))$ | $g(x)$ |
|---|---|---|
| $[0,\infty)$ | $F(x)/(1-F(x))$ | $\frac{f(x)}{(1-F(x))^2} r\left[F(x)/(1-F(x))\right]$ |
| $[0,\infty)$ | $-\log(1-F(x))$ | $\frac{f(x)}{1-F(x)} r\left[-\log(1-F(x))\right]$ |
| $[0,\infty)$ | $F^{\alpha}(x)/(1-F^{\alpha}(x))$ | $\frac{\alpha f(x)F^{\alpha-1}(x)}{(1-F^{\alpha}(x))^2} r\left[F^{\alpha}(x)/(1-F^{\alpha}(x))\right]$ |
| $[0,\infty)$ | $-\log(1-F^{\alpha}(x))$ | $\frac{\alpha f(x)F^{\alpha-1}(x)}{1-F^{\alpha}(x)} r\left[-\log(1-F^{\alpha}(x))\right]$ |
| $(-\infty,+\infty)$ | $\log\left[F(x)/(1-F(x))\right]$ | $\frac{f(x)}{F(x)\left(1-F(x)\right)} r\left[\log\left[F(x)/(1-F(x))\right]\right]$ |
| $(-\infty,+\infty)$ | $\log\left[-\log(1-F(x))\right]$ | $\frac{f(x)}{(F(x)-1)\log\left(1-F(x)\right)} r\left[\log\left[-\log(1-F(x))\right]\right]$ |
| $(-\infty,+\infty)$ | $\log\left[F^{\alpha}(x)/(1-F^{\alpha}(x))\right]$ | $\frac{\alpha f(x)}{F(x)\left(1-F^{\alpha}(x)\right)} r\left[\log\left[F^{\alpha}(x)/(1-F^{\alpha}(x))\right]\right]$ |
| $(-\infty,+\infty)$ | $\log\left[-\log(1-F^{\alpha}(x))\right]$ | $\frac{\alpha f(x)F^{\alpha-1}(x)}{(F^{\alpha}(x)-1)\log\left(1-F^{\alpha}(x)\right)} r\left[\log\left[-\log(1-F^{\alpha}(x))\right]\right]$ |

**Remark 1.4.1** *A few observations regarding the family of distributions are described in (1.14):*

- *The relationship between random variables $X$ and $T$ is given by the fact that $G(x) = R(F(x))$: $X = F^{-1}(T)$. This offers a simple method for simulating random variable*

*$X$, compute $X = F^{-1}(T)$, which has the (c.d.f) $G(x)$, after first simulating random variable $T$ from (p.d.f) $r(t)$. Thus, $\mathbb{E}(X) = \mathbb{E}\{F^{-1}(T)\}$ can be used to obtain $\mathbb{E}(X)$.*

- *The following formula can be used to calculate the quantile function, $Q(p)$, $0 < p < 1$, for the $T - X$ family of distributions*

$$Q(p) = F^{-1}\left\{R^{-1}(p)\right\}.$$

**Theorem 1.4.1** *This theorem provides the connection between the Shannon entropy of the generator, $r(t)$, and the Shannon entropy of the new family of distributions, $g(x)$.*

*So if a random variable $X$ follows the family of distributions (1.14), then the Shannon entropy of $X$, $\eta_X$, is given by:*

$$\eta_X = -\mathbb{E}\left[\log f(F^{-1}(T))\right] - \eta_T. \tag{1.15}$$

*where $\eta_T = -\mathbb{E}\left[\log r(t)\right]$, is the Shannon entropy for the random variable $T$ with (p.d.f) $r(t)$.*

***Proof.*** *By definition we have:*

$$\begin{aligned}\eta_X &= -\mathbb{E}\left[\log g(X)\right],\\ &= -\mathbb{E}\left[\log\left\{f(x)r(F(x))\right\}\right],\\ &= -\mathbb{E}(\log f(x)) - \mathbb{E}\left[\log\left\{r(F(x))\right\}\right].\end{aligned}$$

*The random variable $T = F(x)$ has the (p.d.f) $r(t)$thus, it implies:*

$\mathbb{E}(\log f(x)) = \mathbb{E}\left[\log f(F^{-1}(T))\right]$, $-\mathbb{E}\left[\log\left\{r(F(x))\right\}\right] = -\mathbb{E}\left[\log r(t)\right] = \eta_T$.

*Then finally $\eta_X = -\mathbb{E}\left[\log f(F^{-1}(T))\right] - \eta_T$, which is the result in (1.15).* ■

### 1.4.2 Some beta-X Family When $\mathbf{W(F(x)) = F(x)}$

#### Beta-exponential Family

The statistical distribution that is most frequently used to address reliability issues is the exponential distribution. A generalization of the exponential distribution was proposed by Nadarajah, S. and Kotz [55] in the hopes that it would have a wider range of reliability applications. The following general class serves as inspiration for the generalization: If $F$ represents a random variable's cumulative distribution function ($c.d.f.$), then a generalized class of distributions can be defined by

$$G(x) = I_{F(x)}(a,b) \quad \text{with} \quad a>0,\ b>0, \tag{1.16}$$

$$= \frac{1}{B(a,b)} \int_0^{F(x)} t^{a-1}(1-t)^{b-1} dt. \tag{1.17}$$

Of course, the question of which distribution in(1.16)is the most tractable arises. One of the most fundamental distributions in statistics is the exponential distribution. Thus, by considering $F$ in (1.16) to be the cdf of an exponential distribution with parameter $\theta$, they have been motivated to provide the beta exponential $BE$ distribution. The $BE$ distribution's CDF turns into

$$G(x) = I_{1-\exp(-\theta x)}(a,b) \quad \text{with} \quad a>0,\ b>0, \theta>0,\ \text{and } x\ >0. \tag{1.18}$$

$$G(x) = \frac{B_{1-\exp(-\theta x)}(a,b)}{B(a,b)} \quad \text{with} \quad a>0,\ b>0, \theta>0,\ \text{and } x\ >0. \tag{1.19}$$

The probability density function ($pdf$) that corresponds to this and the hazard rate function linked to (1.18) are:

$$g(x,a,b,\theta) = \frac{\theta}{B(a,b)} \exp(-b\theta x) \left(1-\exp(-\theta x)\right)^{a-1}, \tag{1.20}$$

$$g(x,a,b,\theta) = \frac{\theta\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \exp(-b\theta x) \left(1-\exp(-\theta x)\right)^{a-1}. \tag{1.21}$$

And

$$h(x) = \frac{\theta}{B_{\exp(-\theta x)}(b,a)} \exp(-b\theta x) \left(1-\exp(-\theta x)\right)^{a-1}. \tag{1.22}$$

where $B_{\exp(-\theta x)}(b,a) = \displaystyle\int_0^{\exp(-\theta x)} t^{b-1}(1-t)^{a-1}dt$, denotes the incomplete beta function.

**Proof.** we have $h(x) = \dfrac{g(x)}{1-G(x)}$, then:

$$h(x) = \frac{\frac{\theta}{B(a,b)} \exp(-b\theta x) \left(1-\exp(-\theta x)\right)^{a-1}}{\frac{B(a,b)-B_{1-\exp(-\theta x)}(a,b)}{B(a,b)}},$$

$$= \frac{\theta \exp(-b\theta x) \left(1-\exp(-\theta x)\right)^{a-1}}{B(a,b)-B_{1-\exp(-\theta x)}(a,b)}.$$

we need to calculate this $B(a,b) - B_{1-\exp(-\theta x)}(a,b)$ , so:

$$
\begin{aligned}
B(a,b) - B_{1-\exp(-\theta x)}(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1}\,dt - \int_0^{1-\exp(-\theta x)} t^{a-1}(1-t)^{b-1}\,dt, \\
&= \int_{1-\exp(-\theta x)}^{1} t^{a-1}(1-t)^{b-1}\,dt, \\
&= -\int_{\exp(-\theta x)}^{0} w^{b-1}(1-w)^{a-1}\,dw, \\
&= \int_0^{\exp(-\theta x)} w^{b-1}(1-w)^{a-1}\,dw, \\
&= B_{\exp(-\theta x)}(b,a).
\end{aligned}
$$

Thus,

$$
h(x) = \frac{\theta}{B_{\exp(-\theta x)}(b,a)} \exp(-b\theta x)\left(1-\exp(-\theta x)\right)^{a-1}.
$$

■

Some other particular cases of (1.18) are:

- for $b = n - a + 1$ and integer values of $a$

$$
G(x) = \sum_{i=a}^{n} C_i^n \exp(-(n-i)\theta x)(1-\exp(-\theta x))^i. \tag{1.23}
$$

- for integer values of a:

$$
G(x) = 1 - \frac{\exp(-b\theta x)}{\Gamma(b)} \sum_{i=1}^{a} \frac{\Gamma(b+i-1)}{\Gamma(i)}(1-\exp(-\theta x))^{i-1}. \tag{1.24}
$$

- for integer values of b:

$$
G(x) = \frac{(1-\exp(-\theta x))^a}{\Gamma(a)} \sum_{i=1}^{b} \frac{\Gamma(a+i-1)}{\Gamma(i)} \exp(-\theta x\,(i-1)). \tag{1.25}
$$

- for $a = \frac{1}{2}$ and $b = \frac{1}{2}$ :

$$
G(x) = \frac{2}{\pi} \arctan\left(\sqrt{\exp(-\theta x) - 1}\right). \tag{1.26}
$$

**Shape** Firstly, consider the shapes of (1.20) and (1.22). Note from (1.20) thet $g(x) \sim \frac{\lambda^a x^{a-1}}{B(a,b)}$ as $x \to 0$ and that $g(x) \sim \frac{\lambda e^{-b\lambda x}}{B(a,b)}$ as $x \to \infty$. The first and the second derivatives of $\log(g(x))$ are:

$$\frac{d \log(g(x))}{dx} = \frac{(a-1)\,\lambda e^{-\lambda x}}{1-e^{-\lambda x}} - b\lambda. \tag{1.27}$$

and

$$\frac{d^2 \log(g(x))}{dx^2} = \frac{(1-a)\,\lambda^2 e^{-\lambda x}}{(1-e^{-\lambda x})^2}. \tag{1.28}$$

- If $a < 1$ then $\log(g)'(x)$ is an increasing function with $\log(g)'(\infty) = -b\lambda < 0$. This implies in turn that $f(x)$ monotonically decreases if $a < 1$.
- If $a > 1$ then $\log(g)'(x)$ is an decreasing function with $\log(g)'(0) = \infty$ and $\log(g)'(\infty) = -b\lambda < 0$ This implies that if $a > 1$.
- $f(x)$ has a unique mode at $x = x_0$ with $f(x)$ decreasing for $x > x_0$ and $f(x)$ increasing for $x < x_0$ . So here $x = x_0$ is the root of the equation:

$$\lambda \frac{(a-1)\,e^{-\lambda x}}{1-e^{-\lambda x}} = b\lambda.$$

Note from (1.22) that $\lambda(x) \sim \frac{\lambda^\alpha x^{\alpha-1}}{B(a,b)}$ as $x \to 0$ and that $\lambda(x) \sim b\lambda$ as $x \to \infty$. If $a = 1$ then $\lambda(x)$ is a constant taking the value $b\lambda$. If $a < 1$ then $\lambda(x)$ monotonically decreases with $x$. If $a > 1$ then $\lambda(x)$ monotonically increases with $x$. It's interesting to observe that the forms of $f(x)$ and $\lambda(x)$ are only dependent on parameter $a$, not parameter $b$. The explanation for this can be seen in (1.27) and (1.28). It seems to me to be remarkable for the failure rate features of a family of distributions with two shape parameters to be so simple and to depend on only one of them," as mentioned by Jones [35] in his discussion article. As a family of lifetime distributions, this may be a desirable characteristic of log(beta) distributions. This shows that one might consider employing the exponentiated exponential distribution (special case of (1.20) for $b = 1$) without losing much flexibility.

**Characteristic function** Here, for a random variable $X$ with the pdf (1.20), we construct the characteristic and moment-generating functions.

The definition of its moment-generating function $(mgf)$ :

$$M(t) = \mathbb{E}(\exp(tX)), \tag{1.29}$$
$$= \frac{\theta}{B(a,b)} \int_0^\infty \exp((t-b\theta)x)\,(1-\exp(-\theta x))^{a-1}\,dx.$$

we assume that $y = \exp(-\theta x)$ then we have $dx = -\dfrac{dy}{\theta \exp(-\theta x)}$ and $y \in [0,1]$

$$M(t) = \frac{1}{B(a,b)} \int_0^1 y^{b-\frac{t}{\theta}-1}\,(1-y)^{a-1}\,dy. \tag{1.30}$$

We also know that:

$$\int_0^1 y^{b-\frac{t}{\theta}-1}\,(1-y)^{a-1}\,dy = B(b-\frac{t}{\theta},a).$$

Thus, it is immediate that:

$$M(t) = \frac{B(b-\frac{t}{\theta},a)}{B(a,b)}. \tag{1.31}$$

Hence, the characteristic function of $X$ defined by:

$$Q(t) = \mathbb{E}(\exp(itX)) = \frac{B(b-\frac{it}{\theta},a)}{B(a,b)}. \tag{1.32}$$

where $i = \sqrt{-1}$ is the complex number.

**Moments** From 1.31 the $n$th moment of $X$ can be written as:

$$\frac{(-1)^n}{\theta^n B(a,b)} \frac{d^n}{dp^n} B(a,1+p-a)\,|_{p=a+b-1}\,. \tag{1.33}$$

Particularly, it is possible to calculate the first four moments as:

$$\mathbb{E}(X) = \frac{\Psi\,(a+b) - \Psi\,(b)}{\theta}. \tag{1.34}$$

$$\mathbb{E}(X^2) = \frac{\Psi'\,(b) - \Psi'\,(a+b) - \Psi^2\,(b) - 2\Psi\,(b)\,\Psi\,(a+b) + \Psi^2\,(a+b)}{\theta^2}. \tag{1.35}$$

$$\mathbb{E}(X^3) = \frac{\begin{array}{c}\Psi''\,(a+b) - \Psi''\,(b) - 3\Psi\,(b)\,\Psi'\,(b) + 3\Psi\,(a+b)\,\Psi'\,(b) + 3\Psi'\,(a+b)\,\Psi\,(b) \\ -3\Psi\,(a+b)\,\Psi'\,(a+b) - \Psi^3\,(b) + 3\Psi\,(a+b)\,\Psi^2\,(b) - 3\Psi^2\,(a+b)\,\Psi\,(b) + \Psi^3\,(a+b)\end{array}}{\theta^3}. \tag{1.36}$$

$$\mathbb{E}(X^4) = \frac{\begin{array}{c}\Psi'''(b) - 3\Psi'''(a+b) + 4\Psi(b)\Psi''(b) - 4\Psi(a+b)\Psi''(b) \\ -4\Psi''(a+b)\Psi(b) + 4\Psi(a+b)\Psi''(a+b) + 3\left[\Psi'(b)\right]^2 - 6\Psi'(a+b)\Psi'(b) \\ +3\left[\Psi'(a+b)\right]^2 + 6\Psi^2(b)\Psi'(b) - 12\Psi(b)\Psi(a+b)\Psi'(b) + 6\Psi^2(a+b)\Psi'(b) \\ -6\Psi'(a+b)\Psi^2(b) + 12\Psi(b)\Psi(a+b)\Psi'(a+b) - 6\Psi^2(a+b)\Psi'(a+b) \\ +\Psi^4(b) - 4\Psi(a+b)\Psi^3(b) + 6\Psi^2(a+b)\Psi^2(b) - 4\Psi(b)\Psi^3(a+b) + \Psi^4(a+b)\end{array}}{\theta^4}. \tag{1.37}$$

**Mean deviations** TThe amount of scatter in a population is clearly measured to some extent by the totality of departures from the mean and median. They are referred to as the mean deviation about the mean and the mean deviation about the median, and they are defined by:

$$\delta_1(X) = \int_0^\infty |x-\mu|\, g(x)dx.$$
$$\delta_2(X) = \int_0^\infty |x-M|\, g(x)dx.$$

where $\mu = \mathbb{E}(X)$ and $M$ denotes the median. These measures can be calculated using these relationships:

$$\begin{aligned}\delta_1(X) &= \int_0^\mu (\mu - x)\, g(x)dx + \int_\mu^\infty (x-\mu)\, g(x)dx, \\ &= 2\int_0^\mu (\mu - x)\, g(x)dx, \\ &= 2\left[\mu G(\mu) - \int_0^\mu x g(x)dx\right].\end{aligned} \tag{1.38}$$

and

$$\begin{aligned}\delta_2(X) &= \int_0^M (M - x)\, g(x)dx + \int_M^\infty (x-M)\, g(x)dx, \\ &= 2MG(M) - M - \int_0^M x g(x)dx + \int_M^\infty x g(x)dx, \\ &= \mathbb{E}(X) + 2MG(M) - M - 2\int_0^M x g(x)dx.\end{aligned} \tag{1.39}$$

Using the series representation:

$$(1+z)^a = \sum_{i=0}^\infty \frac{\Gamma(a+1)\, z^i}{\Gamma(a-i+1)\, i!}.$$

So

$$\int_0^m xg(x)dx = \frac{\theta}{B(a,b)} \int_0^m x\exp(-b\theta x)(1-\exp(-\theta x))^{a-1}dx,$$
$$= \frac{\theta}{B(a,b)} \int_0^m x\exp(-b\theta x) \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(a)}{\Gamma(a-i)\, i!} \exp(-i\theta x)\, dx,$$
$$= \frac{\theta\Gamma(a)}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(a-i)\, i!} \int_0^m x\exp(-(b+i)\theta x)\, dx,$$

$$\int_0^m xg(x)dx = \frac{\Gamma(a)}{\theta B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i [1-(1+(b+i)\theta m)\exp(-(b+i)\theta m)]}{\Gamma(a-i)\, i!\, (b+i)^2}. \tag{1.40}$$

Substituting (1.40) into (1.38) and (1.39), one obtains the following expressions for the mean deviations:

$$\delta_1(X) = 2\left[\mu I_{1-\exp(-\theta\mu)}(a,b) - \frac{\Gamma(a)}{\theta B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i [1-(1+(b+i)\theta m)\exp(-(b+i)\theta m)]}{\Gamma(a-i)\, i!\, (b+i)^2}\right].$$

and

$$\delta_2(X) = \frac{\Psi(a+b)\Psi(b)}{\theta} + 2MI_{1-\exp(-\theta M)}(a,b) - M,$$
$$- \frac{2\Gamma(a)}{\theta B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i [1-(1+(b+i)\theta m)\exp(-(b+i)\theta m)]}{\Gamma(a-i)\, i!\, (b+i)^2}.$$

#### Rényi and shannon entropy

**Rényi entropy** The uncertainty's variation is measured by a random variable $X$'s entropy. The definition of Rényi entropy is:

$$I_R(s) = \frac{1}{1-s} \log\left\{\int_{R^+} g^s(x,a,b,\theta)dx\right\}. \tag{1.41}$$

Where $s$(integer) $> 0$ and $s \neq 1$.For the BE distribution, we have:

$$I_R(s) = -\log(\theta) + \frac{1}{1-s} \log\left\{\frac{B(sa-s+1, sb)}{B^s(a,b)}\right\}. \tag{1.42}$$

**Proof.** For the Beta-exponential pdf given by 1.18 :

$$\int_{R^+} g^s(x,a,b,\theta)dx = \int_{R^+} \left(\frac{\theta}{B(a,b)} \exp(-b\theta x)\left(1-\exp(-\theta x)\right)^{a-1}\right)^s dx,$$
$$= \frac{\theta^s}{B^s(a,b)} \int_{R^+} \exp(-bs\theta x)\left(1-\exp(-\theta x)\right)^{(a-1)s} dx.$$

On substituting $y = \exp(-\theta x)$, $dx = \frac{-\exp(-\theta x)}{\theta} dy$, and $y \in [0,1]$

$$\int_{R^+} g^s(x,a,b,\theta)dx = \frac{\theta^{s-1}}{B^s(a,b)} \int_0^1 y^{sa-s}\left(1-y\right)^{sb-1} dy,$$
$$= \frac{\theta^{s-1} B(sa-s+1,sb)}{B^s(a,b)}.$$

Hence, 1.41 takes the expression:

$$I_R(s) = -\log(\theta) + \frac{1}{1-s}\log\left\{\frac{B(sa-s+1,sb)}{B^s(a,b)}\right\}.$$

■

**Shannon entropy** Shannon entropy defined by $\mathbb{E}\left(-\log\left(g(x,a,b,\theta)\right)\right)$ is the particular case of Rényi entropy for $s \uparrow 1$. Limiting $s \uparrow 1$ in 1.42 and using L'Hospital's rule, one obtains:

$$\mathbb{E}\left(-\log\left(g(x,a,b,\theta)\right)\right) = -\log(\theta) + \log\left(B(a,b)\right) + (a+b-1)\,\Psi(a+b) - (a-1)\,\Psi(a) - b\Psi(b).$$

**Asymptotics** If $X_1,...,X_n$ is a random sample from 1.20 and if $\bar{X}$ denotes the sample mean then by central limit theorem with $n \to +\infty$

$$\frac{\sqrt{n}\left(\bar{X}-\mathbb{E}(X)\right)}{\sqrt{\mathrm{Var}(X)}} \to N(0.1). \tag{1.43}$$

Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1,...,X_n)$ and $m_n = \min(X_1,...,X_n)$. Note from 1.20 $1-F(t) \sim \frac{\exp(-b\theta t)}{bB(a,b)}$ as $t \to +\infty$ and that $F(t) \sim \frac{\theta t^a}{aB(a,b)}$. Thus, it can be seen using L'Hospital's rule that:

$$\lim_{t\to+\infty} \frac{1-F(t+\frac{x}{b\theta})}{1-F(t)} = \lim_{t\to+\infty} \frac{\exp\left(-(x+b\theta t)\right)}{\exp(-b\theta t)} = \exp(-x). \tag{1.44}$$

and

$$\lim_{t\to 0}\frac{F(tx)}{F(t)}=\lim_{t\to 0}\frac{(tx)^a}{t^a}=x^a. \tag{1.45}$$

Hence, it follows from Theorem 1.6.2 in [43] that there must be norming constants $a_n>0, b_n>0, c_n>0$ and $d_n>0$ such that $P\left(a_n\left(M_n-b_n\right)\leq x\right)\to\exp\left(-\exp\left(-x\right)\right)$ and $P\left(c_n\left(m_n-d_n\right)\leq x\right)\to 1-\exp\left(-x^a\right)$ as $n\to+\infty$.

## Estimation

**The method of moments** Let $X_1,...,X_n$ be a random sample from 1.20. Using the moments approach, equating $\mathbb{E}\left(X\right)$, $\mathrm{Var}\left(X\right)$ and $\mathbb{E}\left[X-\mathbb{E}\left(X\right)\right]^3$, with the corresponding sample estimates:

$$S_1=\frac{1}{n}\sum_{i=1}^{n}X_i, \tag{1.46}$$

$$S_2=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-S_1\right)^2,\ \text{and}$$

$$S_3=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-S_1\right)^3.$$

respectively, one obtains the system of equations:

$$\begin{cases}\dfrac{\Psi\left(a+b\right)-\Psi\left(b\right)}{\theta}=\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}X_i,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1\\ \dfrac{\Psi'\left(b\right)-\Psi'\left(a+b\right)}{\theta^2}=\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(X_i-S_1\right)^2,\ldots\ldots\ldots(2\\ \dfrac{\Psi''\left(a+b\right)-\Psi''\left(b\right)}{\theta^3}=\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(X_i-S_1\right)^3\ldots\ldots\ldots(3\end{cases}$$

Combining (1) with (2) and (1) with (3), one obtains the equations $\dfrac{\Psi'\left(b\right)-\Psi'\left(a+b\right)}{\left(\Psi\left(b\right)-\Psi\left(a+b\right)\right)^2}=\dfrac{S_2}{S_1^2}$ and $\dfrac{\Psi''\left(b\right)-\Psi''\left(a+b\right)}{\left(\Psi\left(b\right)-\Psi\left(a+b\right)\right)^3}=\dfrac{S_3}{S_1^3}$, which can be solved simultaneously to give estimates for $a$ and $b$. The estimate for $\theta$ can then be obtained directly from (1).

**Maximum likelihood method** The log-likelihood for a random sample $X_1,...,X_n$ from 1.20 is:

$$\mathcal{L}\left(x,a,b,\theta\right)=n\log\left(\theta\right)-n\log\left(B\left(a,b\right)\right)-b\theta\sum_{i=1}^{n}x_i+\left(a-1\right)\sum_{i=1}^{n}\log\left(1-\exp\left(-\theta x_i\right)\right). \tag{1.47}$$

The derivatives of this log-likelihood with respect to $a$, $b$ and $\theta$ are:

$$\frac{d\mathcal{L}(x,a,b,\theta)}{da} = -n\Psi(a) + \Psi(a+b) + \sum_{i=1}^{n}\log\left(1-\exp\left(-\theta x_i\right)\right),$$

$$\frac{d\mathcal{L}(x,a,b,\theta)}{db} = -n\Psi(b) + n\Psi(a+b) - \theta\sum_{i=1}^{n}x_i, \text{ and}$$

$$\frac{d\mathcal{L}(x,a,b,\theta)}{d\theta} = \frac{n}{\theta} - b\sum_{i=1}^{n}x_i + (a-1)\sum_{i=1}^{n}\frac{x_i\exp\left(-\theta x_i\right)}{1-\exp\left(-\theta x_i\right)}.$$

which can be solved simultaneously for $a$, $b$ and $\theta$.

Interval estimates of $(a, b, \theta)$ and hypothesis tests require the Fisher information matrix. It is simple to identify the components of this matrix using the findings of (1.34) and (1.35)). It is acquired,

$$\begin{aligned}
\mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{da^2}\right) &= n\Psi'(a) - n\Psi'(a+b),\\
\mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{dadb}\right) &= \mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{dbda}\right) = -n\Psi'(a+b),\\
\mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{dad\theta}\right) &= \mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{d\theta da}\right) = -\frac{nb}{(a-1)\theta}\left[\Psi(a+b) - \Psi(b+1)\right],\\
&\quad\times(\text{provided } a>1),\\
\mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{db^2}\right) &= n\Psi'(b) - n\Psi'(a+b),\\
\mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{dbd\theta}\right) &= \mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{d\theta db}\right) = \frac{n}{\theta}\left[\Psi(a+b) - \Psi(b)\right], \text{ and}\\
\mathbb{E}\left(-\frac{d^2\mathcal{L}(x,a,b,\theta)}{d\theta^2}\right) &= \frac{n}{\theta^2}\left[1 + \frac{b(a+b-1)}{a-2}\left(\begin{array}{c}\Psi'(b+1) - \Psi'(a+b-1) + \Psi^{2}(b+1)\\ -2\Psi(b+1)\Psi(a+b-1) + \Psi^2(a+b-1)\end{array}\right)\right]\\
&\quad\times((\text{provided } a>2)).
\end{aligned}$$

### The Beta-Lindley Distribution

$F(x)$ is the representation of a random variable $X$'s cumulative distribution function (cdf). As demonstrated by Eugene et al.[22], the (cdf) for a generalized class of distribution for the random variable $X$ is then obtained by applying the inverse CDF of a beta distributed random variable:

$$G(x) = \frac{1}{B(a,b)}\int_0^{F(x)} t^{a-1}(1-t)^{b-1}\,dt, \quad 0<a,b<\infty. \tag{1.48}$$

The probability density function for $G(x)$ is given by:

$$g(x) = \frac{f(x)}{B(a,b)} \left(F(x)\right)^{a-1} \left(1 - F(x)\right)^{b-1}. \tag{1.49}$$

Where $f(x)$ is the (pdf) of the Lindley distribution with parameter $\theta$ :

$$f(x) = \frac{\theta^2}{\theta+1} (1+x) \exp\left(-\theta x\right), \qquad 0 < x, \theta. \tag{1.50}$$

And the corresponding cumulative distribution function (cdf) is:

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} \exp\left(-\theta x\right), \qquad 0 < x, \theta. \tag{1.51}$$

Then we have the (cdf) of the Beta-Lindley distribution:

$$G(x) = \frac{1}{B(a,b)} \int_0^{1-\frac{\theta+1+\theta x}{\theta+1}\exp(-\theta x)} t^{a-1} (1-t)^{b-1}\, dt, \qquad 0 < a, b < \infty. \tag{1.52}$$

According to Cordeiro and Nadarajah[20], this (cdf)1.52 can be defined as follows in terms of the hypergeometric function:

$$G(x) = \frac{\left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp\left(-\theta x\right)\right)^a}{aB(a,b)} \times {}_2F_1\left(a, 1-b, a+1, 1 - \frac{\theta+1+\theta x}{\theta+1} \exp\left(-\theta x\right)\right). \tag{1.53}$$

If the parameter $b > 0$ is real noninteger, we have:

$$G(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp\left(-\theta x\right)\right)^{a+i}}{i!\Gamma(b-i)(a+i)}. \tag{1.54}$$

Th (pdf) of the new distribution is given by:

$$g(x) = \frac{\theta^2 (\theta+1+\theta x)^{b-1} (1+x) \exp\left(-\theta b x\right)}{(\theta+1)^b B(a,b)} \times \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp\left(-\theta x\right)\right)^{a-1}. \tag{1.55}$$

**Lemma 1.4.1** *When $a = b = 1$, the BL in 1.55 reduces to the Lindley distribution in 1.50 with parameter $\theta$.*

*When $b = 1$, the BL in 1.55 reduces to the generalized Lindley distribution GLD proposed by Nadarajah et al.[52].*

*The limit of beta-Lindley density as $x \to \infty$ is 0 and the limit as $x \to 0$ is 0.*

**The reliability function** The reliability function of the beta-Lindley distribution is given by:

$$R(t) = 1 - G(t) = 1 - \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp\left(-\theta x\right)\right)^{a+i}}{i!\Gamma(b-i)(a+i)}. \tag{1.56}$$

**The hazard rate function** The hazard rate function is a crucial parameter that describes living things. In general, it can be understood as the conditional probability of failure if it has made it to time $t$. Given the beta-Lindley random variable, the hazard rate function is:

$$h(t) = \frac{g(t)}{1-G(t)} = \left( \frac{\theta^2 \left(\theta+1+\theta t\right)^{b-1} \left(1+t\right) \exp\left(-\theta b t\right)}{\left(\theta+1\right)^b B(a,b)} \times \left(1 - \frac{\theta+1+\theta t}{\theta+1} \exp\left(-\theta t\right) \right)^{a-1} \right) \tag{1.57}$$

$$\times \left(1 - \frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)} \sum_{i=0}^{\infty} \frac{\left(-1\right)^i \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp\left(-\theta x\right) \right)^{a+i}}{i!\Gamma\left(b-i\right)\left(a+i\right)} \right)^{-1}.$$

**Moments and Generating Function** If $a > 0, b > 0$ are real non integers, the $nth$ moments $\mathbb{E}(X^n)$ of the beta-Lindley distributed random variable $X$ is given as:

$$\mathbb{E}(X^n) = \frac{\Gamma^2\left(a+b\right)}{\theta^n \Gamma\left(b\right)} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{\left(-1\right)^{i+j} \exp\left(\left(b+j\right)\left(\theta+1\right)\right)}{\Gamma\left(a-j\right)\Gamma\left(i-1\right)\Gamma\left(a+b-i\right)\left(\theta+1\right)^{b+j-i} j! \left(b+j\right)^{a+b+n-i}} \tag{1.58}$$

$$\times \left[\theta \Gamma\left(n-i+a+b, \left(b+j\right)\left(\theta+1\right)\right) + \frac{1}{b+j}\Gamma\left(n-i+a+b+1, \left(b+j\right)\left(\theta+1\right)\right)\right].$$

**Proof.** we have

$$\begin{aligned}
\mathbb{E}(X^n) &= \int_0^{+\infty} x^n g(x) dx, \\
&= \frac{\theta}{\left(\theta+1\right)^b B\left(a,b\right)} \int_0^{+\infty} \left(\frac{t}{\theta}\right)^n \left(\theta+1+t\right)^{b-1} \left(\frac{\theta+t}{\theta}\right) \exp\left(-bt\right) \\
&\times \left(1 - \frac{\theta+1+t}{\theta+1}\exp\left(-t\right)\right)^{a-1} dt, \\
&= \frac{\Gamma\left(a\right)}{B\left(a,b\right)\theta^n} \sum_{j=0}^{+\infty} \frac{\left(-1\right)^j}{j!\Gamma\left(a-j\right)\left(\theta+1\right)^{b+j}} \\
&\times \left( \begin{array}{c} \theta \int_0^{+\infty} t^n \left(\theta+1+t\right)^{a+b-1} \exp\left(-\left(b+j\right)t\right) dt \\ + \int_0^{+\infty} t^{n+1} \left(\theta+1+t\right)^{a+b-1} \exp\left(-\left(b+j\right)t\right) dt \end{array} \right), \\
&= \frac{\Gamma\left(a\right)}{B\left(a,b\right)\theta^n} \sum_{j=0}^{+\infty} \frac{\left(-1\right)^j \exp\left(\left(b+j\right)\left(\theta+1\right)\right)}{j!\Gamma\left(a-j\right)\left(\theta+1\right)^{b+j}} \sum_{i=0}^{+\infty} \frac{\left(-1\right)^i \Gamma\left(a+b\right)\left(\theta+1\right)^i}{\Gamma\left(a+b-i\right)\Gamma\left(i-1\right)} \\
&\times \left(\theta \int_{\theta+1}^{+\infty} t^{n+a+b-i-1} \exp\left(-\left(b+j\right)t\right) + \int_{\theta+1}^{+\infty} t^{n+a+b-i} \exp\left(-\left(b+j\right)t\right)\right).
\end{aligned}$$

With

$$\int_{\theta+1}^{+\infty} t^{n+a+b-i-1} \exp\left(-\left(b+j\right)t\right) = \frac{1}{\left(b+j\right)^{n+a+b-i}} \Gamma\left(n+a+b-i, \left(\theta+1\right)\left(b+j\right)\right).$$

And finaly we have:

$$\mathbb{E}(X^n) = \frac{\Gamma^2(a+b)}{\theta^n \Gamma(b)} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j} \exp((b+j)(\theta+1))}{\Gamma(a-j)\Gamma(i-1)\Gamma(a+b-i)(\theta+1)^{b+j-i} j!(b+j)^{a+b+n-i}}$$
$$\times \left[\theta\Gamma(n-i+a+b,(b+j)(\theta+1)) + \frac{1}{b+j}\Gamma(n-i+a+b+1,(b+j)(\theta+1))\right].$$

■

**Order Statistics** The $k$th order statistic of a sample is its $k$th smallest value. And we know that if $X_{(1)} \leq ... \leq X_{(n)}$ denotes the order statistic of a random sample from a continuous population with (cdf) $G(x)$ and (pdf) $g(x)$, then the (pdf) of $X_{(k)}$ is given as:

$$g_{X_{(k)}(x)} = \frac{n!}{(k-1)!(n-k)!} g(x) (G(x))^{k-1} (1-G(x))^{n-k}.$$

So, for $k = 1, .., n$. The (pdf) of the $k$th order statistic for the beta-Lindley distribution is given by:

$$g_{X_{(k)}(x)} = \frac{n!}{(k-1)!(n-k)!} \left( \begin{array}{c} \frac{\theta^2 (\theta+1+\theta x)^{b-1} (1+x) \exp(-\theta b x)}{(\theta+1)^b B(a,b)} \times \\ \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp(-\theta x)\right)^{a-1} \end{array} \right) \tag{1.59}$$
$$\times \left( \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp(-\theta x)\right)^{a+i}}{i!\Gamma(b-i)(a+i)} \right)^{k-1}$$
$$\times \left( 1 - \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \left(1 - \frac{\theta+1+\theta x}{\theta+1} \exp(-\theta x)\right)^{a+i}}{i!\Gamma(b-i)(a+i)} \right)^{n-k}.$$

**Estimation**

**Maximum Likelihood Estimates** The maximum likelihood estimates (MLEs) for the parameters that make up the beta-Lindley distribution function are produced by the following procedures:

So, the likelihood function of the observed sample $\underset{\sim}{x} = \{x_1, x_2, ..., x_n\}$ of size $n$ drawn from the density 1.55 is defined as:

$$L\left(\underset{\sim}{x},a,b,\theta\right)=\prod_{i=1}^{n}g\left(x_i\right), \tag{1.60}$$
$$=\prod_{i=1}^{n}\left(\begin{array}{c}\frac{\theta^2\left(\theta+1+\theta x_i\right)^{b-1}\left(1+x_i\right)\exp\left(-\theta b x_i\right)}{\left(\theta+1\right)^b B(a,b)}\\ \times\left(1-\frac{\theta+1+\theta x_i}{\theta+1}\exp\left(-\theta x_i\right)\right)^{a-1}\end{array}\right),$$
$$=\left[\frac{\theta^2}{\left(\theta+1\right)^b B(a,b)}\right]^n\exp\left(-\theta b\sum_{i=1}^{n}x_i\right)$$
$$\times\prod_{i=1}^{n}\left(1+x_i\right)\left(\theta+1+\theta x_i\right)^{b-1}\left(1-\frac{\theta+1+\theta x_i}{\theta+1}\exp\left(-\theta x_i\right)\right)^{a-1}.$$

And the corresponding log-likelihood function is given by:

$$\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)=\log L\left(\underset{\sim}{x},a,b,\theta\right),$$
$$=n\left(2\log\left(\theta\right)-\log\left(B(a,b)\right)-b\log\left(\theta+1\right)\right)-\theta b\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left(1+x_i\right)$$
$$+(b-1)\sum_{i=1}^{n}\log\left(\theta+1+\theta x_i\right)+(a-1)\sum_{i=1}^{n}\log\left(1-\frac{\theta+1+\theta x_i}{\theta+1}\exp\left(-\theta x_i\right)\right).$$

Now, to obtain the parameters estimates, we need to solve this nonlinear system of equations:

$$\begin{cases}\frac{d\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)}{da}=0,\\ \frac{d\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)}{db}=0,\ \text{and}\\ \frac{d\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)}{d\theta}=0.\end{cases}$$

So, we have

$$\begin{cases}\frac{d\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)}{da}=n\psi\left(a+b\right)-n\psi\left(a\right)+\sum_{i=1}^{n}\log\left(1-\frac{\theta+1+\theta x_i}{\theta+1}\exp\left(-\theta x_i\right)\right)=0,\\ \frac{d\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)}{db}=n\psi\left(a+b\right)-n\psi\left(b\right)-n\log\left(\theta+1\right)+\sum_{i=1}^{n}\log\left(\theta+1+\theta x_i\right)-\theta\sum_{i=1}^{n}x_i=0,\ \text{and}\\ \frac{d\mathcal{L}\left(\underset{\sim}{x},a,b,\theta\right)}{d\theta}=\frac{2n}{\theta}-\frac{nb}{\theta+1}+(b-1)\sum_{i=1}^{n}\frac{1+x_i}{\theta+1+\theta x_i}\\ -b\sum_{i=1}^{n}x_i+(a-1)\sum_{i=1}^{n}\frac{\left(\left(\theta+1\right)\left(\theta+1+\theta x_i\right)-1\right)x_i\exp\left(-\theta x_i\right)}{\left(\theta+1\right)^2\left(1-\frac{\theta+1+\theta x_i}{\theta+1}\exp\left(-\theta x_i\right)\right)}=0.\end{cases}$$

where $\psi(.) = \frac{d \log(\Gamma(.))}{d.} = \frac{\Gamma'(.)}{\Gamma(.)}$ is digamma function.

This nonlinear system of equations can be solved to find the MLEs $\left(\hat{a}, \hat{b}, \hat{\theta}\right)$ of $(a, b, \theta)$, respectively. Generally, it is more convenient to numerically maximize the sample likelihood function given in 1.60 using nonlinear optimization algorithms, like the quasi-Newton algorithm. Using the standard large sample approximation, the MLE $\hat{\rho} = \left(\hat{a}, \hat{b}, \hat{\theta}\right)$ can be roughly regarded as a trivariate normal with mean $\rho$ and variance covariance matrix equal to the inverse of the expected information matrix; that is:

$$\sqrt{n}\left(\hat{\rho} - \rho\right) \longrightarrow N\left(0, nI^{-1}(\rho)\right).$$

where $I^{-1}(\rho)$ is the limiting variance-covariance matrix of $\hat{\rho}$.

The elements of the $3 \times 3$ matrix $I(\rho)$ can be estimated by $I_{ij}(\hat{\rho}) = -l_{\rho_i \rho_j} \mid_{\rho=\hat{\rho}}$,for $i, j \in \{1, 2, 3\}$.

The elements of the Hessian matrix corresponding to the $l$ function, are given as:

$$I_{11} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{da^2} = n\psi'(a+b) - n\psi'(a),$$

$$I_{12} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{dadb} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{dbda} = n\psi'(a+b),$$

$$I_{13} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{dad\theta} = \sum_{i=1}^{n} - \frac{(\theta + 2 + x_i + \theta x_i)\, x_i\theta \exp(-\theta x_i)}{(\theta+1)\left(-\theta - 1 + \theta\exp(-\theta x_i) + \exp(-\theta x_i) + x_i\theta\exp(-\theta x_i)\right)},$$

$$I_{22} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{db^2} = n\psi'(a+b) - n\psi'(b),$$

$$I_{23} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{dbd\theta} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{d\theta db} = \frac{-n}{\theta+1} + \sum_{i=1}^{n} \frac{1+x_i}{1+\theta+\theta x_i} - \sum_{i=1}^{n} x_i, \text{ and}$$

$$I_{33} = \frac{d^2\mathcal{L}\left(\underset{\sim}{x}, a, b, \theta\right)}{d\theta^2} = \frac{-2n}{\theta^2} + \frac{nb}{(\theta+1)^2} - (b-1)\sum_{i=1}^{n}\left(\frac{1+x_i}{1+\theta+\theta x_i}\right)^2$$
$$+ (a+1)\sum_{i=1}^{n} \frac{\left(x_i\theta\exp(-\theta x_i)\left(\theta x_i^2 + \theta^3 x_i^2 + 2\theta^2 x_i^2 - 2 - x_i + \theta^3 x_i + 3\theta^2 x_i + \theta x_i\right)\right)}{(\theta+1)^2\left(-1 - \theta + \theta\exp(-\theta x_i) + \exp(-\theta x_i) + x_i\theta\exp(-\theta x_i)\right)}$$
$$- \sum_{i=1}^{n} \frac{\exp(-2\theta x_i)\,\theta^2 x_i^2\,(\theta + 2 + x_i + \theta x_i)^2}{(\theta+1)^2\left(-1 - \theta + \theta\exp(-\theta x_i) + \exp(-\theta x_i) + x_i\theta\exp(-\theta x_i)\right)}.$$

So the asymptotic confidence interval with significance level $\alpha$ for each parameter$\rho_k$

for $k = 1, 2, 3$ is given by:

$$ACI_{100(1-\alpha)}(\rho) = \left[\hat{\rho}_k \pm z_{\frac{\alpha}{2}} \sqrt{I_{kk}(\hat{\rho})}\right].$$

Where $z_{\frac{\alpha}{2}}$ represents the standard normal distribution's quantile $\left(1 - \frac{\alpha}{2}\right)$. The Hessian matrix, also known as the information matrix, and its inverse, as well as the standard errors and asymptotic confidence intervals, are easily computed using R.

**Least Squares Estimators** This section provides an estimator of the unknown parameters of the beta-Lindley distribution using the regression-based approach, which was originally proposed by Swain et al. [67] to estimate the parameters of beta distributions. It can also be used in a number of other circumstances. A random sample of size $n$ from a distribution function $G(\cdot)$ is represented by $X_1, ..., X_n$, and the ordered sample is represented by $X_{(i)}$, $i = 1, 2, ..., n$. The method that is suggested makes use of the distribution of $G(X_{(i)})$. With respect to a sample size of $n$, we possess

$$\begin{aligned}
\mathbb{E}\left(G(X_{(i)})\right) &= \frac{i}{n+1}, \\
\text{Var}\left(G(X_{(i)})\right) &= \frac{i\,(n-i+1)}{(n+1)^2\,(n+2)}, \quad \text{and} \\
Cov\left(G(X_{(i)}), G(X_{(j)})\right) &= \frac{i\,(n-j+1)}{(n+1)^2\,(n+2)} \quad \text{for } i < j.
\end{aligned}$$

See Johnson et al [34]. Applying the least squares method can be done using the variances and the expectations. To obtain the estimators, we need to minimizing

$$Q = \sum_{i=1}^{n} \left(G(X_{(i)}) - \frac{i}{n+1}\right)^2.$$

with respect to the unknown parameters.

For the BL distribution, the least square estimates (LSEs), $\hat{a}, \hat{b}$ and $\hat{\theta}$ of $a, b$ and $\theta$ are defined as those arguments that minimize the objective function:

$$Q = \sum_{i=1}^{n} \left( \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \left(1 - \frac{\theta+1+\theta x_{(i)}}{\theta+1} \exp\left(-\theta x_{(i)}\right)\right)^{a+i}}{i!\Gamma(b-i)\,(a+i)} - \frac{i}{n+1} \right)^2.$$

The minimum point $\hat{a}, \hat{b}$ and $\hat{\theta}$ can also be given as a solution of the following system of non-linear equations:

$$\frac{dQ}{da} = \frac{dlQ}{db} = \frac{dQ}{d\theta} = 0.$$

# Chapter 2

# Truncated Distribution

To handle the complexities of lifetime analysis, statistical models have developed. There are situations where a phenomenon has a narrow range of applications, which makes the foundational model less applicable. In these cases, truncated distributions are quite useful. These distributions can be thought of as a particular kind of conditional distribution, where certain conditions limit the random variable's range. What is known as a lower-truncated distribution results when the random variable's range is constrained below a certain threshold. On the other hand, the distribution that results when the range is limited above a specified threshold is known as an upper-truncated distribution. The last section, is in accordance with the article "The Truncated New XLindley Distribution with Applications" that was published in the Journal of Computational Analysis and Applications (JoCAAA). The truncated versions of the new-XLindley distribution are presented and their properties are examined in this paper. We use the proposed distribution to analyze three distinct medical data sets in order to show its applicability.

## 2.1 Definition of truncation

With a probability density function $f(x)$ or probability mass function $f(x)$, let $X$ be a random variable. The distribution of $X$ is said to be truncated at the point $X = a$ if all of the values of $X \leq a$ are either rejected or unavailable. Consequently, the probability mass function $g_L(x)$of the distribution truncated at $X = a$ is provided by:

$$\begin{aligned} g_L(x) &= \frac{f(x)}{P(X > a)} = \frac{f(x)}{1 - F(a)}, \qquad X > a, \qquad (2.1) \\ &= \frac{f(x)}{\sum\limits_{x>a} f(x)} \qquad \text{(for discrete)}. \\ &= \frac{f(x)}{\int\limits_{x>a} f(x)dx} \qquad \text{(for continuous)}. \end{aligned}$$

The truncated model's probability density function has been defined by Khan et al. (1983)[37]. In the case of a doubly truncated model, the probability density function can be found by representing the truncation points of $a$ and $b$ on the lower and upper, respectively:

$$g_D(x) = \frac{f(x)}{P(a \leq X \leq b)} = \frac{f(x)}{F(b) - F(a)}, \quad a \leq x \leq b, a < b\ . \qquad (2.2)$$

Notice that in fact $f_T(x)$ is a density:

$$\int_a^b g_T(x)dx = \frac{1}{F(b) - F(a)} \int_a^b f(x)dx = 1. \qquad (2.3)$$

The upper truncated distribution's (pdf) can be found using:

$$g_U(x) = \frac{f(x)}{F(b)}, \quad \text{with} \quad x \leq b. \qquad (2.4)$$

where $F(x)$ represents the cumulative distribution function of $X$.

A probability distribution can be derived from a given distribution by shifting the probability mass from outside to inside of a specified interval. Assume that a probability distribution on the line is described by the distribution function $G$. The truncated distribution corresponding to $G$ is the definition of the distribution function.

$$G(x) = \begin{cases} 0 & \text{for} \quad x < a, \\ \dfrac{F(x) - F(a)}{F(b) - F(a)} & \text{for} \quad a \leq x \leq b, \\ 1 & \text{for} \quad x > b, \quad a < b. \end{cases} \qquad (2.5)$$

The double truncated distribution's $r$th moments are provided by

$$\begin{aligned}\mathbb{E}\left[X^r\right] &= \sum_a^b x^r g_D(x) \quad \text{for discrete distribution,}\\ \mathbb{E}\left[X^r\right] &= \int_a^b x^r g_D(x)dx \quad \text{for continuous distribution.}\end{aligned}$$

## 2.2 Censoring and Truncation

Analyzing time-to-event data presents unique challenges because the data can take on various forms. Certain survival studies may include two characteristics.

The first kind, **Censoring**, is frequently observed in survival data. In this section, several types of censorship will be covered. As we shall see, various kinds of censoring schemes, on the left as well as the right, result in distinct likelihood functions.

The second characteristic that is frequently observed in survival data is **Truncation**.

### 2.2.1 Truncated data

It was discovered by Sir Francis Galton (1897), a pioneer in the field of modern statistics, through an analysis of the registered speeds of American trotting horses. Samples that have been truncated are categorized based on whether their terminal points are known or unknown. These points become additional factors to be estimated from sample data when they are unknown. When only individuals whose event time falls inside an observational range $(Y_L, Y_U)$ are visible, survival information is truncated.

#### Upper truncated data

When $Y_L = 0$, the upper truncation happens. That is, we record the survival time $X$ only in the cases where $X \leq Y_U$.

Occasionally, right truncation originates from:

- When estimating their distribution from Earth, stars that are too far away are said to be invisible and right-truncated.
- When a mortality study is conducted using death records.

**Lower truncated data**

Only those people whose event time $X$ surpasses the truncation point $Y_L$ are shown here, unless $Y_U$ is infinite. In this kind of truncation, no individuals are shown who experience the event of interest before the truncation time. Because we only see topics from this point on until they pass away or are censored, it is frequently referred to as a postponed entry time.

### 2.2.2 Censored data

Three things can lead to censorship:

1. The incident doesn't happen prior to "the study ends."
2. A research subject is "lost to follow-up" when the study is underway.
3. A study participant leaves the research.

The experience of multiple participants over time is depicted in FiG.3.1.

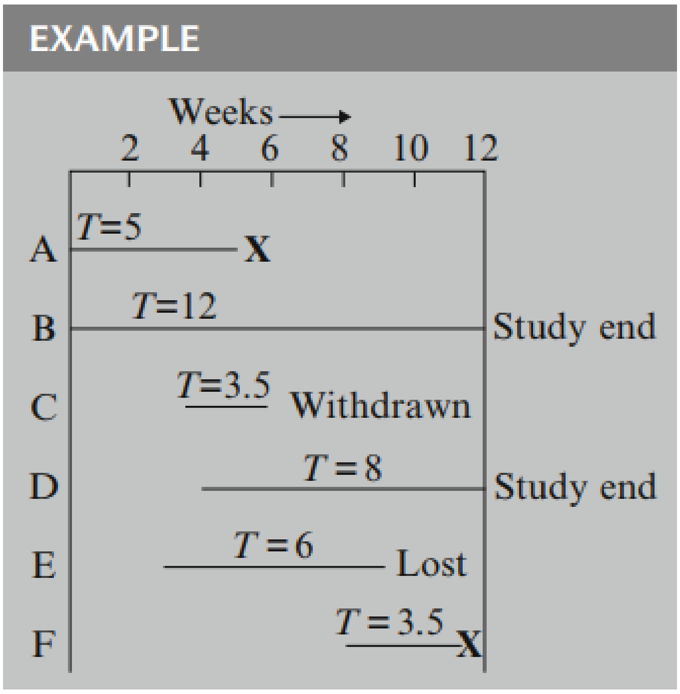


Figure 2.1: $X \Rightarrow$Events occurs

For:

- Participant A: His five-week survival period, which he attained by being monitored from the start of the trial until the fifth week, when the incident happened, is not thought to be legitimate.

- Participant B: The survival period in this instance is only censored for a minimum of 12 weeks because it was also observed from the beginning of the inquiry until the conclusion of the 12th week.

- The second and third participant, C, joins the study in the second and third weeks and is monitored until he leaves in the sixth week; after 3.5 weeks, his survival time is censored.

- Participant D: This participant's censored period is 8 weeks since they entered the study in the fourth week and were monitored for the rest of the study without receiving the event.

- Participant E: He starts the study in week three and is monitored until week nine, at which point he is lost to follow-up before the failure time is reached; his censored time is six weeks.

- Participant F: After eight weeks of observation, they receive the event at week 11.5. Similar to participant A, there is no censorship in this instance; the survivor period is 3.5 weeks.

Here is the table (2.1) with survival time data for each of the six participants shown. We have demonstrated whether or not the event was censored at each instance. With 1 indicating failure and 0 censored. This simplified table illustrates the type of data that will be examined in a survival analysis.

Table 2.1: The type of data analyzed

| Participant | Survival time | Event occurs |
|---|---|---|
| A | 5 | 1 |
| B | 12 | 0 |
| C | 3.5 | 0 |
| D | 8 | 0 |
| E | 6 | 0 |
| F | 3.5 | 1 |

## 2.3 Distribution of Truncated order statistics

### 2.3.1 Lower truncated

The probability density function of $X_{(r)}$

$$g_{X_{(r)}}(x) = C_n^r \left[\frac{F(x_{(r)}) - F(a)}{1 - F(a)}\right]^{r-1} \left[\frac{1 - F\left(x_{(r)}\right)}{1 - F(a)}\right]^{n-r} \frac{f\left(x_{(r)}\right)}{1 - F(a)}. \tag{2.6}$$

### 2.3.2 Upper truncated

The probability density function of $X_{(r)}$

$$g_{X_{(r)}}(x) = C_n^r \left[\frac{F(x_{(r)})}{F(b)}\right]^{r-1} \left[\frac{F(b) - F\left(x_{(r)}\right)}{F(b)}\right]^{n-r} \frac{f\left(x_{(r)}\right)}{F(b)}. \tag{2.7}$$

## 2.4 The Truncated New-XLindley Distributions

In the interval $[a, b]$, a distribution $G(x, \Theta)$ is considered double truncated if its cumulative distribution function (cdf) is defined as follows:

$$G(x, \Theta) = \frac{F(x, \Theta) - F(a, \Theta)}{F(b, \Theta) - F(a, \Theta)}, \qquad a \leq x \leq b, \qquad -\infty < a < b < +\infty. \tag{2.8}$$

And the corresponding probability density function (pdf) is

$$g(x,\Theta)=\frac{f(x,\Theta)}{F(b,\Theta)-F(a,\Theta)},\qquad a\le x\le b,\qquad -\infty<a<b<+\infty. \tag{2.9}$$

where $\Theta\in\Re^n$ indicates the baseline model's vector parameter, and $g(x,\Theta)$ and $G(x,\Theta)$ are the baseline model's (pdf) and (cdf). Three instances in this situation are identifiable as:

- The baseline model is reached when $a=0$ and $b\to+\infty$.
- It is referred to as the baseline model's upper truncated distribution when $a=0$.
- The lower truncated distribution of the baseline model is what happens when $b\to+\infty$.

The baseline model used in this study is the New-XLindley distribution (NXLD) [38], which has the following (cdf) distribution function:

$$F(x)=1-\left(\frac{\theta x}{2}+1\right)\exp(-x\theta),\qquad 0<\theta,x. \tag{2.10}$$

And the corresponding (pdf) is given by:

$$f(x)=\frac{\theta}{2}(\theta x+1)\exp(-x\theta),\qquad 0<\theta,x. \tag{2.11}$$

So, the double truncated new-XLindley distribution is defined as:

$$g_D(x)=\frac{\theta}{2}\frac{(\theta x+1)}{F(b)-F(a)}\exp(-x\theta),\qquad 0\le a\le x\le b<+\infty. \tag{2.12}$$

Only the properties of the upper truncated new-XLindley distribution will be covered in the parts that follow. The same methodology may be used to examine the properties of the double truncated new-XLindley distribution and the lower truncated new-XLindley distribution. The following (pdf) represents the upper truncated new-XLindley distribution:

$$g_U(x)=\frac{\theta(\theta x+1)}{2(e^{b\theta}-1)-b\theta}\exp(-(x-b)\theta),\qquad 0\le x\le b<+\infty. \tag{2.13}$$

And the corresponding (cdf) is given by:

$$G_U(x) = \frac{F(x)}{F(b)} = \frac{\left(2e^{\theta x} - \theta x - 2\right)}{\left(2e^{bx} - bx - 2\right)} \exp(-(x-b)\theta), \qquad 0 \leq x \leq b < +\infty. \tag{2.14}$$

UTNXLD is the designation for it. Take note that the aforementioned (pdf) will act as follows:

$$\frac{dg_U(x)}{dx} = \frac{\theta^3 x}{\theta b - 2\left(e^{\theta b} - 1\right)} \exp(-(x-b)\theta). \tag{2.15}$$

The expression $\theta b - 2\left(e^{\theta b} - 1\right)$ is negative for all $\theta b > 0$, meaning that the derivative is negative for all $x > 0$, then $G_U(x)$ is decreasing over $[0, b]$.

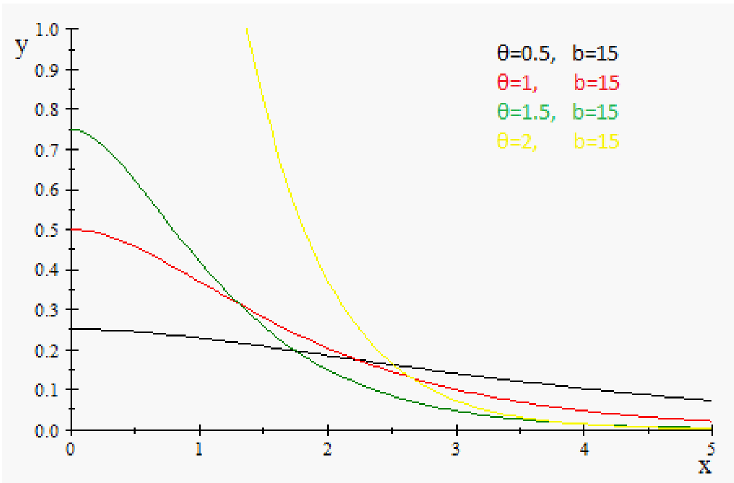


FiG 3.2: The density function of UTNXL distribution for differents $\theta$

The corresponding hazard function at epoch is given by:

$$H(t) = \frac{f(t)}{F(b) - F(t)} = \frac{\theta}{2} \frac{(\theta t + 1)}{F(b) - F(t)} \exp(-(t-b)\theta), \qquad 0 \leq t \leq b < +\infty. \tag{2.16}$$

[47] used the term $\eta(t) = -\dfrac{f'(t)}{f(t)}$ to determine the monotonicity of the hazard function. For UTNXL distribution, we get:

$$\eta_{UTNXL}(t) = -\frac{g'(t)}{g(t)} = -\frac{g'(t)/F(b)}{g(t)/F(b)} = -\frac{f'(t)}{f(t)} = \eta_{NXL}(t). \tag{2.17}$$

It followed that

- $H(0) = \dfrac{\theta}{2\left(e^{b\theta} - 1\right) - b\theta} \exp(-2\theta b).$

- $H(t) \to \infty$,when $t \to b$.
- $\eta_{NXL}(t) = -\dfrac{f'(t)}{f(t)} = \dfrac{\theta^2 t}{1+\theta t} > 0$, it implies that the hazard rate function of UTNXL distribution is increasing in $x$ and $\theta$, see FiG 3.3.

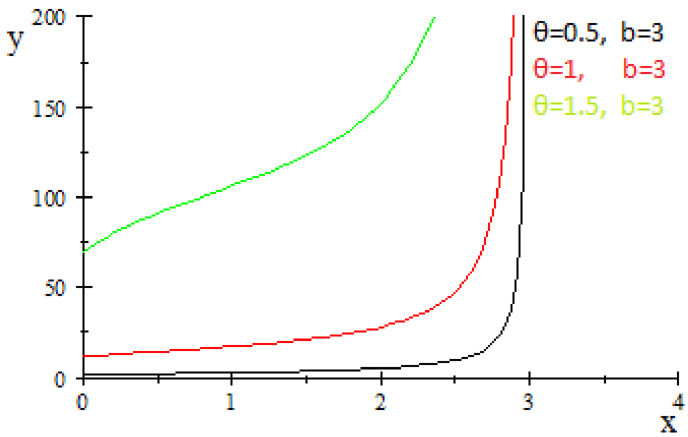


FiG 3.3: The hazard function of UTNXL distribution

for differents $\theta$

**Moments and Related Measures**

The following is the definition of the $r$th moments under the upper truncated New-XLindley distribution:

$$\begin{aligned} E\left[X^r\right] &= \frac{\theta}{2F(b)}\left(\int_0^b x^r (1+\theta x)\, e^{-\theta x} dx\right), \\ &= \frac{\theta}{2F(b)}\left(\int_0^b x^r e^{-\theta x} dx + \theta \int_0^b x^{r+1} e^{-\theta x} dx\right), \\ &= \frac{1}{2\theta^r F(b)}\left(\int_0^{\theta b} y^r e^{-y} dy + \theta \int_0^{\theta b} y^{r+1} e^{-y} dy\right). \end{aligned} \tag{2.18}$$

And using the lower incomplete gamma function defied by:

$$\gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx = (s-1)!\left(1 - e^{-t}\sum_{k=0}^{s-1}\frac{x^k}{k!}\right), \tag{2.19}$$

then we have

$$E\left[X^r\right] = \frac{\gamma(r+1,\theta b) + \gamma(r+2,\theta b)}{2\theta^r F(b)}. \tag{2.20}$$

In particular, the first two moments can be worked out as

$$
\begin{aligned}
E[X] &= \frac{\gamma(2,\theta b)+\gamma(3,\theta b)}{2\theta F(b)} = \frac{3-\left(3(1+\theta b)+(\theta b)^2\right)e^{-\theta b}}{2\theta F(b)}, \\
E\left[X^2\right] &= \frac{\gamma(3,\theta b)+\gamma(4,\theta b)}{2\theta^2 F(b)} = \frac{5\left(1-\left(1+\theta b+\frac{(\theta b)^2}{2}+\frac{(\theta b)^3}{10}\right)e^{-\theta b}\right)}{2\theta^2 F(b)}.
\end{aligned}
\tag{2.21}
$$

The variance is

$$
\delta^2 = E\left[X^2\right]-(E[X])^2. \tag{2.22}
$$

### 2.4.1 Order Statistics

In this subsection, we derive the (pdf) $g_{X_{(j)}(t)}$ of the $(j=1,...,n)$ order statistics $X_{(j)}$ :

$$
g_{X_{(j)}(t)} = \frac{1}{B(j,n-j+1)} g(t)G(t)^{j-1}(1-G(t))^{n-j}, \tag{2.23}
$$

Where $B(j,n-j+1)$ is the beta function. Expanding the binomial expansion, we get:

$$
g_{X_{(j)}(t)} = \frac{1}{B(j,n-j+1)} \sum_{i=0}^{n-j} (-1)^i C_{n-j}^i \left(\frac{F(t)}{F(b)}\right)^{j+i} \frac{f(t)}{F(t)}. \tag{2.24}
$$

where $f(t)$ and $F(t)$ are the New-XLindley distribution's (pdf) and (cdf), respectively. The (pdf) of $X_{(n)}$ id for $j=n$, is provided by:

$$
g_{X_{(j)}(t)} = \frac{n\theta\left(1-\left(\frac{\theta t}{2}+1\right)\exp(-\theta t)\right)^{n-1}(1+\theta t)\exp(-\theta t)}{2\left(1-\left(\frac{\theta b}{2}+1\right)\exp(-\theta b)\right)^n}. \tag{2.25}
$$

Similarly, the pdf of $X_{(1)}$ is given by:

$$
g_{X_{(1)}(t)} = \frac{n\theta}{2}\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i \frac{\left(1-\left(\frac{\theta t}{2}+1\right)\exp(-\theta t)\right)^i(1+\theta t)\exp(-\theta t)}{\left(1-\left(\frac{\theta b}{2}+1\right)\exp(-\theta b)\right)^{i+1}}. \tag{2.26}
$$

### 2.4.2 Quantile Function

Let $X$ be a (UTNXLD), The quantile function $Q(p)=G^{-1}(p)=x_p$.For that we need to solve this equation :

$$
G(x_p)=p, \qquad p\in(0,1). \tag{2.27}
$$

Then we have

$$-\left(\theta x_p+2\right)e^{-(\theta x_p+2)}=2e^{-2}\left(pF\left(b\right)-1\right). \tag{2.28}$$

In [51], random variables with the Lindley or Poisson-Lindley distribution were first created using the Lambert $W$ function to solve the aforementioned equation for $x_p$. A multivalued complex function, the Lambert W function is the answer to the following equation:

$$W\left(z\right)e^{W(z)}=z, \qquad z\in\mathbb{C}. \tag{2.29}$$

We obtained

$$-\left(\theta x_p+2\right)=W_{-1}\left(2e^{-2}\left(pF(b)-1\right)\right). \tag{2.30}$$

where $W_{-1}$ is the Lambert function's negative branch. Next, we have UTNXLD's quantile function.

$$x_p=Q(p)=-\frac{2}{\theta}-\frac{1}{\theta}W_{-1}\left(2e^{-2}\left(pF(b)-1\right)\right). \tag{2.31}$$

As $b\to+\infty$, we get the quantile function of new-XLindley distribution derived by [51]:

$$x_p=Q(p)=-\frac{2}{\theta}-\frac{1}{\theta}W_{-1}\left(2e^{-2}\left(p-1\right)\right). \tag{2.32}$$

The UTNXL distribution's median can be found as follows:

$$x_{med}=Q(1/2)=-\frac{2}{\theta}-\frac{1}{\theta}W_{-1}\left(e^{-2}\left(F(b)-2\right)\right). \tag{2.33}$$

### 2.4.3 Maximum Likelihood Estimation

This section describes how to obtain the lower truncated New-XLindley (LTNXLD) and double truncated New-XLindley (DTNXLD) distributions and the maximum likelihood estimates (MLE) of the UTNXL parameters using a random sample $x=\{x_1,x_2,...,x_n\}$ of size $n$. These distributions can be used to model the real issues, depending on the kind of data. We fitted these distributions to two actual datasets in the section that follows.

#### MLEs for UTNXLD

Let $x=\{x_1,x_2,...,x_n\}$ be a sample of size $n$ that is (iid) independent and $x$ identically distributed in the UTNXL distribution. The log-likelihood function for the given sample is given by:

$$\log L(\theta, b, x) = n \log(\theta) - n \log\left(2\left(e^{\theta b} - 1\right) - \theta b\right) + \sum_{i=1}^{n} \log(1 + \theta x_i) - \theta \sum_{i=1}^{n} (x_i - b).$$

Because $b$ is independent of $x$ in the log-likelihood equation above, estimates of $b$ in terms of the observed sample are not attainable. For each $x_1, x_2, ..., x_n$, let $x_{(1)} < x_{(2)} < ... < x_{(n)}$ be the order sample. The MLE $\hat{b}$ of $b$ is $\hat{b} = \max(x_1, x_2, ..., x_n) = x_{(n)}$, , and the MLE $\hat{\theta}$ of $\theta$ may be found by solving the following non-linear equation:

$$\frac{n}{\theta} - \frac{n\hat{b}\left(2e^{\hat{b}\theta} - 1\right)}{2\left(e^{\hat{b}\theta} - 1\right) - \theta\hat{b}} + \sum_{i=1}^{n} \frac{x_i}{1 + \theta x_i} - \sum_{i=1}^{n} \left(x_i - \hat{b}\right) = 0. \tag{2.34}$$

We must use an iterative procedure akin to Newton's approach in order to address the previously noted issue.

**MLEs for LTNXLD**

The (pdf) of LTNXLD is given by:

$$g_L(x) = \frac{\theta(1 + \theta x)}{2 + \theta a} e^{-\theta(x-a)}.$$

So, the log-likelihood function based on from LTNXL distribution is given by:

$$\log L(\theta, a, x) = n \log(\theta) - n \log(2 + \theta a) + \sum_{i=1}^{n} \log(1 + \theta x_i) - \theta \sum_{i=1}^{n} (x_i - a).$$

Similarly, $\hat{a} = \min(x_1, x_2, ..., x_n) = x_{(1)}$, the smallest observation, will be the greatest likelihood estimate of $a$ from the aforementioned sections. The following non-linear equation can be solved in a unique way to determine the greatest likelihood estimate $\hat{\theta}$ of $\theta$:

$$\frac{n}{\theta} - \frac{n\hat{a}}{2 + \theta\hat{a}} + \sum_{i=1}^{n} \frac{x_i}{1 + \theta x_i} - \sum_{i=1}^{n} (x_i - \hat{a}) = 0. \tag{2.35}$$

The solution to the previously described issue requires the use of an iterative procedure akin to Newton's method.

**MLEs for DTNXLD**

The log-likelihood function under the assumption of the double truncated New-XLindley distribution for $X$ the random variable , is given by:

$$\log L(\theta, a, b, x) = n\log(2) - n\log(\Xi(\theta)) + \sum_{i=1}^{n}\log(1+\theta x_i) - \theta\sum_{i=1}^{n}x_i.$$

Where $\Xi(\theta) = \left(be^{-\theta b} - ae^{-\theta b}\right) + \frac{2}{\theta}\left(e^{-\theta b} - e^{-\theta a}\right)$ , and the MLEs of $a$ and $b$ are $\hat{a} = x_{(1)}$ and $\hat{b} = x_{(n)}$ respectively, the MLE of $\theta$ can be obtained by solving the following non-linear equation :

$$-\frac{n\Xi'(\theta)}{\Xi(\theta)} + \sum_{i=1}^{n}\frac{x_i}{1+\theta x_i} - \sum_{i=1}^{n}x_i = 0. \tag{2.36}$$

### 2.4.4 Real Data Applications

In this part, we use two real data sets to show that the truncated new-XLindley distribution can be a superior model than the Lindley, new-XLindley, and exponential distributions. Additionally, we take into account criteria such as AICC (Consistent Akaike Information Criterion), HQIC (Hannan-Quinn Information Criterion), -2L (-2Log-Likelihood), BIC (Bayesian Information Criterion), and AIC (Akaike Information Criterion) for the data set in order to compare a number of distribution models. Smaller values for AIC, BIC, -2L, AICC, and HQIC indicate a better distribution.

Dataset 1: this data on recovered times (in week) of 74 Angola individuals infected with the Marburg virus (see https://www.who.int/): 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,

2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7,

7, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9.

Dataset 2: Based on 18 determinations by various labs, a numeric vector shows the quantity (percentage of the stated total weight) of shrimp in shrimp cocktail.

Dataset 3: A numeric vector of 15measurements by different laboratories of the pesticide DDT in kale, in ppm (parts per million) using the multiple pesticide residue measurement.(Finsterwalder, 1976).

The data are: 2.79, 2.93, 3.22, 3.78, 3.22, 3.38, 3.18, 3.33, 3.34, 3.06, 3.07, 3.56, 3.08, 4.64, 3.34.

Table 2.2: The ML estimates, -2 log-likelihood, AIC, BIC,AICC, and HQIC for dataset 1

| $distribution$ | $Estimates$ | $AIC$ | $BIC$ | $-2L$ | $CAIC$ | $HQIC$ |
|---|---|---|---|---|---|---|
| $exp(\theta)$ | 0.1989 | 388.9947 | 391.2987 | 386.9947 | 389.0502 | 389.9138 |
| $Lindley(\theta)$ | 0.3466 | 367.9865 | 370.2906 | 365.9865 | 368.0421 | 368.9056 |
| $XLindley(\theta)$ | 0.3128 | 374.8971 | 377.2012 | 372.8971 | 374.9527 | 375.8162 |
| $New-XL(\theta)$ | 0.3111 | 377.825 | 380.129 | 375.825 | 377.8805 | 378.7441 |
| $Zeghdoudi(\theta)$ | 0.5536 | 351.3284 | 353.6325 | 349.3284 | 351.384 | 352.2476 |
| $UTNXLD(\theta, b)$ | $(0.00057, 9)$ | 329.9532 | 334.5613 | 325.9532 | 330.1222 | 331.7915 |
| $LTNXLD(\theta, a)$ | $(0.4410, 2)$ | 317.2225 | 321.8306 | 313.2225 | 317.3915 | 319.0608 |
| $DTNXLD(\theta, a, b)$ | $(0.5035, 2, 9)$ | 328.2239 | 332.832 | 324.2239 | 328.3929 | 330.0621 |

Table 2.3: The ML estimates,-2 log-likelihood, AIC, BIC,AICC, and HQIC for dataset 2

| $distribution$ | $Estimates$ | $AIC$ | $BIC$ | $-2L$ | $CAIC$ | $HQIC$ |
|---|---|---|---|---|---|---|
| $exp(\theta)$ | 0.03144 | 162.5345 | 163.4249 | 160.5345 | 162.7845 | 162.6573 |
| $Lindley(\theta)$ | 0.06109 | 149.7339 | 150.6242 | 147.7339 | 149.9839 | 149.8566 |
| $XLindley(\theta)$ | 0.05943 | 150.6912 | 151.5816 | 148.6912 | 150.9412 | 150.814 |
| $New-XL(\theta)$ | 0.05087 | 157.7884 | 158.6788 | 155.7884 | 158.0384 | 157.9112 |
| $Zeghdoudi(\theta)$ | 0.09296 | 141.4851 | 142.3755 | 139.4851 | 141.7351 | 141.6079 |
| $UTNXLD(\theta, b)$ | 0.00046 | 133.303 | 135.0838 | 129.303 | 134.103 | 133.5486 |
| $LTNXLD(\theta, a)$ | 0.21758 | 99.04227 | 100.823 | 95.04227 | 99.84227 | 99.28781 |
| $DTNXLD(\theta, a, b)$ | 0.26181 | 105.8118 | 108.4829 | 99.81177 | 107.5261 | 106.1801 |

It is revealed that the proposed upper truncated, lower truncated, and double truncated distributions has lower values of AIC, BIC, log-likelihood, HQIC and ACIC, the goodness of-fit measure.

Table 2.4: The ML estimates,-2 log-likelihood, AIC, BIC,AICC, and HQIC for dataset 3

| $distribution$ | $Estimates$ | $AIC$ | $BIC$ | $-2L$ | $CAIC$ | $HQIC$ |
|---|---|---|---|---|---|---|
| $exp(\theta)$ | 0.3004 | 68.07115 | 68.7792 | 66.07115 | 68.37884 | 68.0636 |
| $Lindley(\theta)$ | 0.5007 | 61.84888 | 62.55693 | 59.84888 | 62.15658 | 61.84134 |
| $XLindley(\theta)$ | 0.44029 | 64.57162 | 65.27967 | 62.57162 | 64.87932 | 64.56408 |
| $New-XL(\theta)$ | 0.4854 | 64.18241 | 64.89046 | 62.18241 | 64.4901 | 64.17486 |
| $Zeghdoudi(\theta)$ | 0.81449 | 53.14825 | 53.8563 | 51.14825 | 53.45595 | 53.14071 |
| $UTNXLD(\theta,b)$ | 0.0006 | 50.12314 | 51.53924 | 46.12314 | 51.12314 | 50.10805 |
| $LTNXLD(\theta,a)$ | 2.0968 | 15.30998 | 16.72608 | 11.30998 | 16.30998 | 15.2949 |
| $DTNXLD(\theta,a,b)$ | 2.2577 | 18.09554 | 20.21969 | 12.09554 | 20.27736 | 18.07291 |

# Chapter 3

# Beta new-XLindley distribution

This chapter presents a novel continuous probability distribution known as the beta-new XLindley distribution, which extends the new-XLindley distribution. Various mathematical properties of this distribution are explored, including the moment-generating function, the moment, entropy, stress-strength reliability, and order statistics. The unknown parameters associated with the beta-new XLindley distribution are estimated using several methods. To demonstrate the practicality of the proposed model, we apply it to analyze two different data sets related to medical data. We intend to capture researchers' attention and showcase this new distribution's versatility and potential applications. This chapter is related to the papers "Beta new-XLindley distribution Distribution with Applications" submitted to the web science journal and "Two-Parameter Beta-Exponential Distribution: Properties and Applications in Demography and Geo-standards" published in the MAS Journal of Applied Sciences.

## 3.1 Distribution model formulation

Let $F(x)$ denote the cumulative distribution function of a random variable $X$, and then the cumulative distribution function for a generalized class of distribution for the random variable $X$, as defied by Eugene et al [22],is generated by applying the inverse $(cdf)$ to a beta distributed random variable to obtain

$$G(x) = \frac{1}{\beta(a,b)} \int_0^{F(x)} t^{a-1}(1-t)^{b-1}dt \quad \text{where } 0 < a, b < +\infty. \tag{3.1}$$

The corresponding probability density function for $G(x)$ is given by

$$g(x) = \frac{1}{\beta(a,b)} f(x)[F(x)]^{a-1}[1-F(x)]^{b-1} \quad \text{where } 0 < a, b < +\infty. \tag{3.2}$$

Where $g(x) = \dfrac{dG(x)}{dx}$

We now introduce the three-parameter beta-new XLindley (BNXL) distribution by taking $G_{BNXL}(x)$ to be the (*cdf*) of $F_{NXL}(x,\theta)$.

The (*cdf*) of the BNXL distribution is then

$$\begin{aligned} G_{BNXL}(x) &= I_{F_{NXL}(x,\theta)}(x), \\ &= \frac{1}{\beta(a,b)} \int_0^{F_{NXL}(x,\theta)} t^{a-1}(1-t)^{b-1}dt, \\ &= \frac{1}{\beta(a,b)} \int_0^{1-(\frac{x\theta}{2}+1)\exp(-x\theta)} t^{a-1}(1-t)^{b-1}dt. \end{aligned} \tag{3.3}$$

This (*cdf*) can be expressed in terms of the hypergeometric function (see Cordeiro and Nadarajah [18] ) in the following way:

$$G_{BNXL}(x) = \frac{1}{\beta(a,b)} \times \frac{F_{NXL}(x,\theta)}{a} \times_2 F_1(a, 1-b, a+1, F_{NXL}(x,\theta)). \tag{3.4}$$

Here, we provide simple expansions for the BNXL distribution (*cdf*) which depends on whether the parameter $b$ (or $a$) is real non-integer or integer. We consider the series expansion

$$(1-z)^{b-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b)}{k!\Gamma(b-k)} z^k. \tag{3.5}$$

valid for $|z| < 1$ and $b > 0$ real non-integer, then we have:

$$G_{BNXL}(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k [F_{NXL}(x,\theta)]^{a+k}}{k!(a+k)\Gamma(b-k)}. \tag{3.6}$$

Thus, we have for $b > k$

$$G_{BNXL}(x) = \frac{\Gamma(a+b)[1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k[1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^k}{k!(a+k)\Gamma(b-k)}. \tag{3.7}$$

The (*pdf*) of the BNXLD is given by:

$$g_{BNXL}(x) = \frac{\left(\frac{1}{2}\right)^b \theta\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (x\theta+1)(x\theta+2)^{b-1} [1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^{a-1} \exp(-xb\theta). \tag{3.8}$$

The hazard rate function for the BNXLD random variable is given by:

$$h(t) = \left( \frac{\left(\frac{1}{2}\right)^b \theta\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (x\theta+1)(x\theta+2)^{b-1} [1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^{a-1} \exp(-xb\theta) \right) \times \left( 1 - \frac{\Gamma(a+b)[1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^a}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k[1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^k}{k!(a+k)\Gamma(b-k)} \right)^{-1}.$$

**Remark 3.1.1** *If $a = b = 1$, the (pdf) of the BNXL distribution reduces to the new-XLindley distribution with parameter $\theta$.*

**Proposition 3.1.1** *The PDF $g_{BNXL}(x)$ in (3.8) of the BNXL distribution is :*

1. Decreasing if $a \leq 1$.

2. Unimodel if $a > 1$.

**Proof.** The first derivative of the *pdf* in (3.8) is determined as follows

$$\frac{dg_{BNXL}(x)}{dx} = 2\left(\frac{1}{2}\right)^{a+b} e^{-bx\theta} \frac{\theta^2\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{(x\theta+2)^{b-2}}{(2e^{-x\theta}+x\theta e^{-x\theta}-2)^2} \left(2 - x\theta e^{-x\theta} - 2e^{-x\theta}\right)^a L(x),$$

where $L(x) = P_0(x) + P_1(x)e^{-x\theta}$ for $P_0(x) = 2 - 2b - 2bx^2\theta^2 - 4bx\theta$ and $P_1(x) = (2a+2b-4) + (5a+5b-6)\theta x + \theta^2(4a+4b-1)x^2 + \theta^3(a+b-1)x^3$.

We can see that $\frac{dg_{BNXL}(x)}{dx}$ and $L(x)$ have the same sign because $\left(2 - x\theta e^{-x\theta} - 2e^{-x\theta}\right) \in [0, 2]$.

If $a \leq 1$, we can ckeck that $L(x) < 0$, which $g_{BNXL}(x)$ is decreasing.

If $a > 1$, we use intermediate value theorem for interval $[0, +\infty[$and $L(0)L(+\infty) < 0$, which exists a real number $x_0 \in [0, +\infty[$ such that $L(x_0) = 0$ also, we have $\frac{d^2 g_{BNXL}(x)}{dx^2} < 0$,

then, there is the unique critical point which maximize the *pdf* (3.8) and the *pdf* is uni-model. Therefore, the mode of BNXL distribution can be found using numerical methods. The verification of the above proposition can also be seen in Figure **??** where the green *pdf* curve is decreasing for $a \leq 1$ and the remaining are uni-model for $a > 1$.

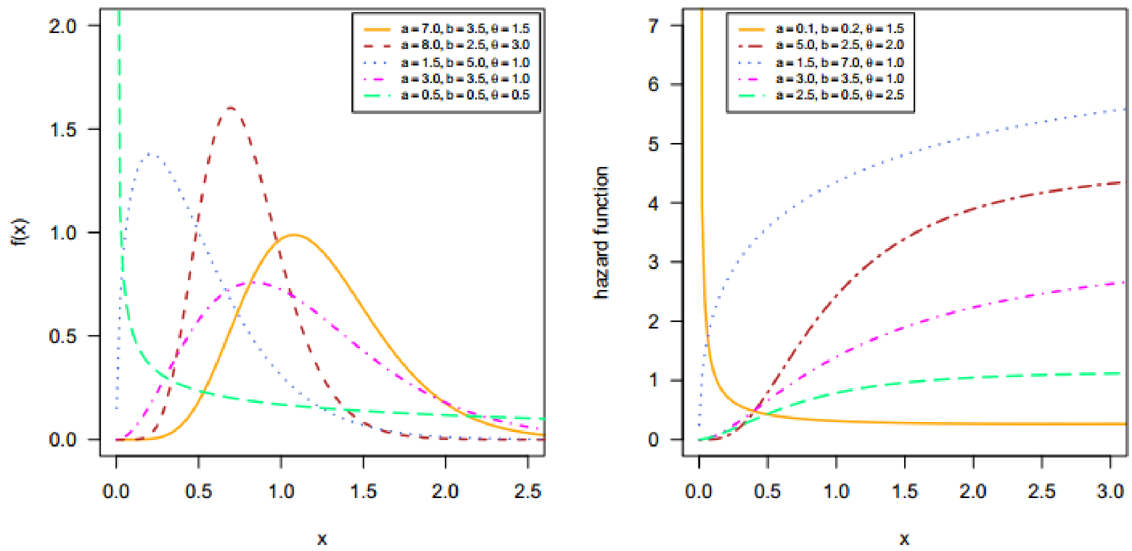


FiG 4.1: Various shap es of p df and hrf of BNXL

■

**Proposition 3.1.2** *The density function of the BNXL distribution can be represented in a linear form as:*

$$g_{BNXL}(x) = W_{j,i,r,s}\left(\theta x^{s+1} + x^{s}\right)\exp(-x\theta\,(r+1)) \quad \textit{with} \quad 0 \leq \theta, x \tag{3.9}$$

**Proof.** We have

$$g_{BNXL}(x) = \frac{\theta}{2B(a,b)}\,(x\theta+1)\left(\frac{x\theta}{2}+1\right)^{b-1}[1-(\frac{x\theta}{2}+1)\exp(-x\theta)]^{a-1}\exp(-xb\theta). \tag{3.10}$$

We regard the series expansion as valid for $|\ z\ | < 1$ and $a > 0$ real non-integer:

$$(1-z)^{a-1} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{j!\Gamma(a-j)}z^{j}. \tag{3.11}$$

Then, we have

$$\left[1-(\frac{x\theta}{2}+1)\exp(-x\theta)\right]^{a-1} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{j!\Gamma(a-j)}\left(\frac{x\theta}{2}+1\right)^{j}\exp(-xj\theta), \tag{3.12}$$

$$g_{BNXL}(x) = \frac{\theta}{2B(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{j!\Gamma(a-j)}\,(x\theta+1)\left(\left(\frac{x\theta}{2}+1\right)\exp(-x\theta)\right)^{b+j-1}\exp(-x\theta). \tag{3.13}$$

Also, we have

$$\left(1-\left(1-\left(\frac{x\theta}{2}+1\right)\exp(-x\theta)\right)\right)^{b+j-1}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b+j)}{i!\Gamma(b+j-i)}\left(1-\left(\frac{x\theta}{2}+1\right)\exp\left(-x\theta\right)\right)^{i},$$

and

$$\begin{aligned}\left(1-\left(\frac{x\theta}{2}+1\right)\exp(-x\theta)\right)^{i} &= \sum_{r=0}^{i}\frac{(-1)^r\Gamma(i+1)}{r!\Gamma(i+1-r)}\left(\left(\frac{x\theta}{2}+1\right)\exp(-x\theta)\right)^{r}\\ &= \sum_{r=0}^{i}(-1)^r C_r^i\left(\frac{x\theta}{2}+1\right)^{r}\exp(-xr\theta).\end{aligned}$$

Then, we have

$$\left(1-\left(1-\left(\frac{x\theta}{2}+1\right)\exp\left(-x\theta\right)\right)\right)^{b+j-1}=\sum_{i=0}^{\infty}\sum_{r=0}^{i}C_r^i\frac{(-1)^{i+r}\Gamma(b+j)}{i!\Gamma(b+j-i)}\left(\frac{x\theta}{2}+1\right)^{r}\exp(-x\theta r).$$

$$g_{BNXL}(x)=\left\{\begin{array}{c}\dfrac{\theta}{2B(a,b)}\displaystyle\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{r=0}^{i}C_r^i\frac{(-1)^{j+i+r}\Gamma(b+j)\Gamma(a)}{i!\Gamma(b+j-i)j!\Gamma(a-j)}\\ \times\left(x\theta+1\right)\left(\dfrac{x\theta}{2}+1\right)^{r}\exp(-x\theta r)\exp(-x\theta)\end{array}\right. . \tag{3.14}$$

By using the binomial expansion for $\left(1+\dfrac{x\theta}{2}\right)^{r}$

$$\left(1+\frac{x\theta}{2}\right)^{r}=\sum_{s=0}^{r}C_s^r\left(\frac{x\theta}{2}\right)^{s},$$

$$g_{BNXL}(x)=\left\{\begin{array}{c}\dfrac{1}{B(a,b)}\displaystyle\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{r=0}^{i}\sum_{s=0}^{r}\left(\frac{\theta}{2}\right)^{s+1}C_s^rC_r^i\\ \times\dfrac{(-1)^{j+i+r}\Gamma(b+j)\Gamma(a)}{i!\Gamma(b+j-i)j!\Gamma(a-j)}\left(\theta x^{s+1}+x^s\right)\exp(-x\theta\left(r+1\right))\end{array}\right. . \tag{3.15}$$

We take

$$W_{j,i,r,s}=\frac{1}{B(a,b)}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{r=0}^{i}\sum_{s=0}^{r}\left(\frac{\theta}{2}\right)^{s+1}C_s^rC_r^i\frac{(-1)^{j+i+r}\Gamma(b+j)\Gamma(a)}{i!\Gamma(b+j-i)j!\Gamma(a-j)}. \tag{3.16}$$

Finally, the pdf of BNXL distribution is defined as

$$g_{BNXL}(x)=W_{j,i,r,s}\left(\theta x^{s+1}+x^s\right)\exp(-x\theta\left(r+1\right)). \tag{3.17}$$

■

## 3.2 Statistical properties

### 3.2.1 Moments

**Theorem 3.2.1** *If $X \sim BNXL(a, b, \theta)$, then the kth moment is given by:*

$$\mathbb{E}(X^k) = W_{j,i,r,s} \frac{1}{\theta^k (r+1)^{k+s+1}} \left[ \frac{1}{r+1} \Gamma(k+s+2) + \Gamma(k+s+1) \right]. \tag{3.18}$$

**Proof.** By using Proposition 4.1.2, we have

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k g_{BNXL}(x) dx = W_{j,i,r,s} \int_0^\infty \left( \theta x^{k+s+1} + x^{k+s} \right) \exp(-x\theta (r+1)) dx \\ &= W_{j,i,r,s} \left[ \theta \int_0^\infty x^{k+s+1} \exp(-x\theta (r+1)) dx + \int_0^\infty x^{k+s} \exp(-x\theta (r+1)) dx \right]. \end{aligned}$$

By taking $x\theta (r+1) = v$, so $x = \frac{v}{\theta (r+1)}$, $dx = \frac{dv}{\theta (r+1)}$

$$E(X^k) = \frac{W_{j,i,r,s}}{\theta^{k+s+1} (r+1)^{k+s+1}} \left[ \frac{1}{r+1} \int_0^\infty v^{k+s+1} \exp(-v) dv + \int_0^\infty v^{k+s} \exp(-v) dv \right]$$

.Finally, we have

$$E(X^k) = \frac{W_{j,i,r,s}}{\theta^k (r+1)^{k+s+1}} \left[ \frac{1}{r+1} \Gamma(k+s+2) + \Gamma(k+s+1) \right].$$

■

**Remark 3.2.1** *In particular, the first four moments can be worked out as*

$$\begin{aligned} \mathbb{E}(X) &= W_{j,i,r,s} \frac{1}{\theta (r+1)^{s+2}} \left[ \frac{1}{r+1} \Gamma(s+3) + \Gamma(s+2) \right], \\ \mathbb{E}(X^2) &= W_{j,i,r,s} \frac{1}{\theta (r+1)^{s+3}} \left[ \frac{1}{r+1} \Gamma(s+4) + \Gamma(s+3) \right], \\ \mathbb{E}(X^3) &= W_{j,i,r,s} \frac{1}{\theta (r+1)^{s+4}} \left[ \frac{1}{r+1} \Gamma(s+5) + \Gamma(s+4) \right], \\ \mathbb{E}(X^4) &= W_{j,i,r,s} \frac{1}{\theta (r+1)^{s+5}} \left[ \frac{1}{r+1} \Gamma(s+6) + \Gamma(s+5) \right]. \end{aligned}$$

### 3.2.2 Moments generating function

Suppose $X$ is a random variable with density function $g(x)$, We now derive an explicit expression for the **mgf**, say $M(s) = \mathbb{E}(\exp(sX))$ of $X$.

$$M(s) = \mathbb{E}(\exp(sX)) = \int_0^{\infty} \exp(sx)g(x)dx. \tag{3.19}$$

By the same manner used in Proposition (4.2), we have

$$M(s) = W_{j,i,r,l} \left[ \frac{\theta}{(\theta r + \theta - s)^{l+2}} \Gamma(l+2) + \frac{1}{(\theta r + \theta - s)^{l+1}} \Gamma(l+1) \right]. \tag{3.20}$$

### 3.2.3 Entropy

It is commonly accepted that the degree of uncertainty in a probability distribution may be determined using entropy and information. However, a lot of correlations have been made using entropy's properties. The uncertainty variation is measured by the entropy of a random variable, $X$. The following is a definition of Rényi's entropy:

$$I_R(s) = \frac{1}{1-s} \log \left\{ \int_{R+} f^s(x)dx \right\}, \tag{3.21}$$

where $s(integer) > 0$ and $s \neq 1$.

**Proposition 3.2.1** *We can write $g^s(x)$ like this*

$$g^s(x) = W_{j,i,r,l,m} (x\theta)^{m+l} \exp(-x\theta(r+s)), \tag{3.22}$$

*where*

$$\begin{aligned} W_{j,i,r,l,m} &= \frac{\theta^s}{2^s [B(a,b)]^s} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^{i} \sum_{l=0}^{r} \sum_{m=0}^{s} C_l^r C_m^s C_r^i \\ &\times \frac{(-1)^{j+i+r} \Gamma((b-1)s+j+1)\Gamma((a-1)s+1)}{i!\Gamma((b-1)s+j+1-i)j!\Gamma((a-1)s+1-j)} \left(\frac{1}{2}\right)^l. \end{aligned} \tag{3.23}$$

**Proof.** The proof is omitted because it is similar to Proposition 4.1.2. ■

**Theorem 3.2.2** *The Rényi's entropy of the BNXL distribution is given by*

$$I_R(s) = \frac{1}{1-s} \log \left\{ \frac{W_{j,i,r,l,m}}{\theta(r+s)^{m+l+1}} \Gamma(m+2) \right\}. \tag{3.24}$$

**Proof.** By using Proposition 4.2.1, we have

$$I_R(s) = \frac{1}{1-s}\log\left\{\int_{R^+} g^s(x)dx\right\} = \frac{1}{1-s}\log\left\{\int_{R^+} W_{j,i,r,l,m}\,(x\theta)^{m+l}\exp(-x\theta\,(r+s))dx\right\}$$

$$= \frac{1}{1-s}\log\left\{W_{j,i,r,l,m}\int_{R^+}(x\theta)^{m+l}\exp(-x\theta\,(r+s))dx\right\},$$

we need to calculate this $\int_{R^+}(x\theta)^{m+l}\exp(-x\theta\,(r+s))dx$, for this we take $x\theta\,(r+s) = v$ so $x = \frac{v}{\theta\,(r+s)}$, $dx = \frac{dv}{\theta\,(r+s)}$

$$\begin{aligned}\int_{R^+}(x\theta)^{m+l}\exp(-x\theta\,(r+s))dx &= \frac{1}{\theta\,(r+s)^{m+l+1}}\int_{R^+} v^{m+l}\exp(-v)dv \\ &= \frac{1}{\theta\,(r+s)^{m+l+1}}\Gamma\,(m+2)\,.\end{aligned}$$

Now, the Rényi entropy for the BNXL distribution is determined as follows

$$I_R(s) = \frac{1}{1-s}\log\left\{\frac{W_{j,i,r,l,m}}{\theta\,(r+s)^{m+l+1}}\Gamma\,(m+2)\right\}.$$

■

## 3.3 Simulation experiment

The objective of the simulation experiment is to assess the performance of various estimation methods in estimating the parameters of the BNXL distribution. We have executed an exhaustive simulation experiment within this segment to evaluate various estimation methods: MLE, ADE, CVME, MPSE, LSE, RTADE, WLSE, MSADE, and MSALDE. During this experiment, we generated sample sizes of 35, 100, 250, 400, and 600 from a pool of 1000 Monte Carlo samples. Tables 1 to 6 provide insights into the bias (BIAS), mean square error (MSE), mean residual error (MRE), and rank values associated with each estimation method. These measures were computed using the following expressions

- BIAS: The average difference between the estimated and true parameters, $\hat{\theta}$ and $\theta$, is used to compute the bias: BIAS$=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i-\theta\right)$.

- MSE: The average of the squared discrepancies between the true and estimated values of the parameter $\theta$ is known as the MSE: MSE=$\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2$.
- MRE: The mean of the absolute differences between the true and estimated parameters, $\hat{\theta}$ and $\theta$, is known as the MRE: MRE=$\frac{1}{N}\sum_{i=1}^{N}\left|\hat{\theta}_i - \theta\right|^2$.

where $N$ is the number of simulated data sets, $\hat{\theta}_i$ is the estimated parameter in th $i^{th}$ data set and $\theta$ is the true parameter value. A discernible trend is observed across these tables, indicating a gradual decrease in Bias, MSE, and MRE toward zero as the sample size expands. This outcome substantiates the consistent estimability of parameters characterizing the BNXL distribution.

## 3.4 Real Data Applications

Compared to the exponential, Lindley, XLindley, new XLindley, Xgamma, Zeghdoudi, two-parameter Lindley I, two-parameter Lindley II, gamma Lindley, new quasi Lindley, power XLindley, beta-Lindley, beta-exponential, and Chen distributions, we show in this section that the beta-new XLindley distribution is better at modeling. Two real-world data sets are used in this evaluation. We use metrics like AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), -2L (-2 Log-Likelihood), and AICC (Consistent Akaike Information Criterion) for every data set in order to assess the efficacy of different distribution models. It is thought that the distribution model with lower AIC, BIC, -2L, and AICC values is better.

The first data set represents the regular time series giving the luteinizing hormone in blood samples at 10 mins intervals from a human female, 48 samples reported by Diggle [21]. The second one represents the recovered times (in a week) of 74 Angola individuals infected with the Marburg virus; see the following link for its information https://www.who.int/.

The variance-covariance matrix $I^{-1}(\hat{\lambda})$ of the MLEs under the beta-new XLindley distribution for data set 1 is computed as

$$\begin{pmatrix} 141.64163 & -399.45164 & 27.74016 \\ -399.45164 & 1256.84388 & -86.16419 \\ 27.74016 & -86.16419 & 5.916374 \end{pmatrix}.$$

The variances of the MLE of $a$, $b$, and $\theta$ under the beta-new XLindley distribution for data set 1 are $var\,(\hat{a}) = 141.64163$, $var\left(\hat{b}\right) = 1256.84388$, and $var\left(\hat{\theta}\right) = 5.916374$. Thus, 95% confidence intervals for $a$, $b$, and $\theta$ are $[0, 40.00921]$, $[0, 76.52157]$, and $[0, 5.574792]$, respectively.

The variance-covariance matrix $I^{-1}(\hat{\lambda})$ of the MLEs under the beta-new XLindley distribution for data set 2 is computed as

$$\begin{pmatrix} 6426.206462 & -1.010469 & 42.351445 \\ -1.010469 & 0.0003792497 & -0.0101268704 \\ 42.351445 & -0.0101268704 & 0.37801906 \end{pmatrix}.$$

The variances of the MLE of $a$, $b$, and $\theta$ under the beta-new XLindley distribution for data set 2 are $var\,(\hat{a}) = 6426.206462$, $var\left(\hat{b}\right) = 0.0003792497$, and $var\left(\hat{\theta}\right) = 0.37801906$. Thus, 95% confidence intervals for $a$, $b$, and $\theta$ are $[0, 256.5861]$, $[0.04780428, 0.1241423]$, and $[2.574209, 4.984309]$, respectively.

Tables 3.7 and 3.8 show parameter MLEs to these fitted distributions for data set 1 and 2, and these Tables show the values of $-2L$, $AIC$, $BIC$, and $AICC$. Thus, the values in these tables indicate that the beta-new XLindley is a stronger competitor than the other distributions.

The likelihood ratio (LR) test is a test of hypothesis in which two different maximum likelihood estimates of a parameter are compared to decide whether to reject or not to reject a restriction on the parameter. The null hypothesis $H_0 : a = b = 1$ versus $H_1 : a \neq 1 \vee b \neq 1$ and $\lambda_{LR} = 2\left(l(x, \hat{\Theta}) - l(x, \Theta_0)\right)$, where $\hat{\Theta}$ and $\Theta_0$ are the MLEs under $H_1$ and $H_0$, respectively. The likelihood ratio test values for the two real data sets are 91.49962 and 41.2729, and their P-values are 0.0 and 1.090696e-09, respectively.

Table 3.1: BIAS, MSE and MRE for (a = 1.5, b = 0.5, theta= 0.25).

| n | Est. | Est. Par. | MLE | ADE | CVME | MPSE | LSE | RTADE | WLSE | MSADE | MSALDE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 35 | BIAS | $\hat{a}$ | $0.38907^{\{7\}}$ | $0.38869^{\{6\}}$ | $0.40419^{\{9\}}$ | $0.39846^{\{8\}}$ | $0.38188^{\{3\}}$ | $0.3844^{\{4\}}$ | $0.38111^{\{2\}}$ | $0.38495^{\{5\}}$ | $0.36899^{\{1\}}$ |
| | | $\hat{b}$ | $0.17434^{\{5\}}$ | $0.17496^{\{6\}}$ | $0.17124^{\{3\}}$ | $0.17247^{\{4\}}$ | $0.17819^{\{8\}}$ | $0.16978^{\{2\}}$ | $0.16929^{\{1\}}$ | $0.18243^{\{9\}}$ | $0.17517^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.09065^{\{9\}}$ | $0.08752^{\{4\}}$ | $0.08674^{\{3\}}$ | $0.08841^{\{6\}}$ | $0.08914^{\{7\}}$ | $0.08669^{\{2\}}$ | $0.08785^{\{5\}}$ | $0.09004^{\{8\}}$ | $0.08591^{\{1\}}$ |
| | MSE | $\hat{a}$ | $0.26235^{\{7\}}$ | $0.25696^{\{5\}}$ | $0.28022^{\{9\}}$ | $0.2751^{\{8\}}$ | $0.25209^{\{4\}}$ | $0.25197^{\{3\}}$ | $0.24924^{\{2\}}$ | $0.25908^{\{6\}}$ | $0.23804^{\{1\}}$ |
| | | $\hat{b}$ | $0.04296^{\{5\}}$ | $0.04435^{\{7\}}$ | $0.042^{\{3\}}$ | $0.04206^{\{4\}}$ | $0.04516^{\{8\}}$ | $0.04135^{\{2\}}$ | $0.04097^{\{1\}}$ | $0.04681^{\{9\}}$ | $0.04379^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.01258^{\{8\}}$ | $0.01199^{\{5\}}$ | $0.01172^{\{2\}}$ | $0.01211^{\{6\}}$ | $0.01246^{\{7\}}$ | $0.01179^{\{3\}}$ | $0.01194^{\{4\}}$ | $0.01261^{\{9\}}$ | $0.01152^{\{1\}}$ |
| | MRE | $\hat{a}$ | $0.25938^{\{7\}}$ | $0.25913^{\{6\}}$ | $0.26946^{\{9\}}$ | $0.26564^{\{8\}}$ | $0.25459^{\{3\}}$ | $0.25627^{\{4\}}$ | $0.25407^{\{2\}}$ | $0.25664^{\{5\}}$ | $0.24599^{\{1\}}$ |
| | | $\hat{b}$ | $0.34867^{\{5\}}$ | $0.34992^{\{6\}}$ | $0.34248^{\{3\}}$ | $0.34494^{\{4\}}$ | $0.35639^{\{8\}}$ | $0.33957^{\{2\}}$ | $0.33858^{\{1\}}$ | $0.36487^{\{9\}}$ | $0.35033^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.36261^{\{9\}}$ | $0.35008^{\{4\}}$ | $0.34694^{\{3\}}$ | $0.35366^{\{6\}}$ | $0.35657^{\{7\}}$ | $0.34675^{\{2\}}$ | $0.35138^{\{5\}}$ | $0.36015^{\{8\}}$ | $0.34365^{\{1\}}$ |
| | $\sum Ranks$ | | $62^{\{8\}}$ | $49^{\{5\}}$ | $44^{\{4\}}$ | $54^{\{6\}}$ | $55^{\{7\}}$ | $24^{\{2\}}$ | $23^{\{1\}}$ | $68^{\{9\}}$ | $26^{\{3\}}$ |
| 100 | BIAS | $\hat{a}$ | $0.23603^{\{5\}}$ | $0.24925^{\{8\}}$ | $0.23013^{\{1\}}$ | $0.25015^{\{9\}}$ | $0.23532^{\{3\}}$ | $0.24489^{\{6\}}$ | $0.24638^{\{7\}}$ | $0.23525^{\{2\}}$ | $0.23591^{\{4\}}$ |
| | | $\hat{b}$ | $0.17064^{\{5\}}$ | $0.17119^{\{6\}}$ | $0.16954^{\{3\}}$ | $0.17018^{\{4\}}$ | $0.17338^{\{8\}}$ | $0.16793^{\{1\}}$ | $0.16804^{\{2\}}$ | $0.17716^{\{9\}}$ | $0.17192^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.08665^{\{8\}}$ | $0.08494^{\{4\}}$ | $0.084^{\{2\}}$ | $0.0855^{\{5\}}$ | $0.08375^{\{1\}}$ | $0.08572^{\{6\}}$ | $0.08613^{\{7\}}$ | $0.08734^{\{9\}}$ | $0.08472^{\{3\}}$ |
| | MSE | $\hat{a}$ | $0.10417^{\{5\}}$ | $0.1121^{\{7\}}$ | $0.09423^{\{2\}}$ | $0.12072^{\{9\}}$ | $0.0939^{\{1\}}$ | $0.11008^{\{6\}}$ | $0.11709^{\{8\}}$ | $0.10152^{\{3\}}$ | $0.10368^{\{4\}}$ |
| | | $\hat{b}$ | $0.04066^{\{3\}}$ | $0.04153^{\{5\}}$ | $0.04088^{\{4\}}$ | $0.04157^{\{6\}}$ | $0.04258^{\{8\}}$ | $0.04049^{\{2\}}$ | $0.03956^{\{1\}}$ | $0.04395^{\{9\}}$ | $0.04225^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.01198^{\{8\}}$ | $0.01126^{\{3\}}$ | $0.01115^{\{2\}}$ | $0.01151^{\{6\}}$ | $0.01086^{\{1\}}$ | $0.01174^{\{7\}}$ | $0.01131^{\{4.5\}}$ | $0.012^{\{9\}}$ | $0.01131^{\{4.5\}}$ |
| | MRE | $\hat{a}$ | $0.15736^{\{5\}}$ | $0.16616^{\{8\}}$ | $0.15342^{\{1\}}$ | $0.16676^{\{9\}}$ | $0.15688^{\{3\}}$ | $0.16326^{\{6\}}$ | $0.16426^{\{7\}}$ | $0.15683^{\{2\}}$ | $0.15727^{\{4\}}$ |
| | | $\hat{b}$ | $0.34127^{\{5\}}$ | $0.34238^{\{6\}}$ | $0.33908^{\{3\}}$ | $0.34036^{\{4\}}$ | $0.34676^{\{8\}}$ | $0.33586^{\{1\}}$ | $0.33607^{\{2\}}$ | $0.35433^{\{9\}}$ | $0.34384^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.34661^{\{8\}}$ | $0.33976^{\{4\}}$ | $0.33599^{\{2\}}$ | $0.34199^{\{5\}}$ | $0.33501^{\{1\}}$ | $0.34289^{\{6\}}$ | $0.34453^{\{7\}}$ | $0.34936^{\{9\}}$ | $0.3389^{\{3\}}$ |
| | $\sum Ranks$ | | $52^{\{7\}}$ | $51^{\{6\}}$ | $20^{\{1\}}$ | $57^{\{8\}}$ | $34^{\{2\}}$ | $41^{\{3\}}$ | $45.5^{\{5\}}$ | $61^{\{9\}}$ | $43.5^{\{4\}}$ |
| 250 | BIAS | $\hat{a}$ | $0.14873^{\{6\}}$ | $0.14257^{\{3\}}$ | $0.14935^{\{7\}}$ | $0.14763^{\{5\}}$ | $0.15409^{\{9\}}$ | $0.14308^{\{4\}}$ | $0.13606^{\{1\}}$ | $0.15046^{\{8\}}$ | $0.13672^{\{2\}}$ |
| | | $\hat{b}$ | $0.1692^{\{7\}}$ | $0.16925^{\{8\}}$ | $0.16786^{\{5\}}$ | $0.16733^{\{3\}}$ | $0.17181^{\{9\}}$ | $0.16242^{\{2\}}$ | $0.16168^{\{1\}}$ | $0.16818^{\{6\}}$ | $0.16734^{\{4\}}$ |
| | | $\hat{\theta}$ | $0.0828^{\{5\}}$ | $0.08378^{\{9\}}$ | $0.08121^{\{3\}}$ | $0.08293^{\{6\}}$ | $0.08311^{\{7\}}$ | $0.07999^{\{2\}}$ | $0.08128^{\{4\}}$ | $0.08341^{\{8\}}$ | $0.07887^{\{1\}}$ |
| | MSE | $\hat{a}$ | $0.03809^{\{5\}}$ | $0.03538^{\{3\}}$ | $0.03982^{\{7\}}$ | $0.03891^{\{6\}}$ | $0.04138^{\{9\}}$ | $0.03545^{\{4\}}$ | $0.03151^{\{1\}}$ | $0.03997^{\{8\}}$ | $0.03212^{\{2\}}$ |
| | | $\hat{b}$ | $0.04046^{\{6\}}$ | $0.04094^{\{8\}}$ | $0.03967^{\{3\}}$ | $0.04014^{\{4\}}$ | $0.04235^{\{9\}}$ | $0.03806^{\{2\}}$ | $0.03745^{\{1\}}$ | $0.04045^{\{5\}}$ | $0.04082^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.01076^{\{6\}}$ | $0.01101^{\{8\}}$ | $0.01032^{\{2\}}$ | $0.0108^{\{7\}}$ | $0.01072^{\{4\}}$ | $0.01033^{\{3\}}$ | $0.01075^{\{5\}}$ | $0.01113^{\{9\}}$ | $0.01013^{\{1\}}$ |
| | MRE | $\hat{a}$ | $0.09915^{\{6\}}$ | $0.09505^{\{3\}}$ | $0.09956^{\{7\}}$ | $0.09842^{\{5\}}$ | $0.10272^{\{9\}}$ | $0.09538^{\{4\}}$ | $0.09071^{\{1\}}$ | $0.10031^{\{8\}}$ | $0.09115^{\{2\}}$ |
| | | $\hat{b}$ | $0.33839^{\{7\}}$ | $0.33849^{\{8\}}$ | $0.33572^{\{5\}}$ | $0.33465^{\{3\}}$ | $0.34363^{\{9\}}$ | $0.32484^{\{2\}}$ | $0.32335^{\{1\}}$ | $0.33637^{\{6\}}$ | $0.33468^{\{4\}}$ |
| | | $\hat{\theta}$ | $0.33119^{\{5\}}$ | $0.33511^{\{9\}}$ | $0.32484^{\{3\}}$ | $0.33172^{\{6\}}$ | $0.33245^{\{7\}}$ | $0.31996^{\{2\}}$ | $0.32512^{\{4\}}$ | $0.33364^{\{8\}}$ | $0.31547^{\{1\}}$ |
| | $\sum Ranks$ | | $53^{\{6\}}$ | $59^{\{7\}}$ | $42^{\{4\}}$ | $45^{\{5\}}$ | $72^{\{9\}}$ | $25^{\{3\}}$ | $19^{\{1\}}$ | $66^{\{8\}}$ | $24^{\{2\}}$ |
| 400 | BIAS | $\hat{a}$ | $0.11617^{\{3\}}$ | $0.11789^{\{6\}}$ | $0.11335^{\{1\}}$ | $0.11635^{\{4\}}$ | $0.1191^{\{8\}}$ | $0.11909^{\{7\}}$ | $0.11553^{\{2\}}$ | $0.11722^{\{5\}}$ | $0.11991^{\{9\}}$ |
| | | $\hat{b}$ | $0.15942^{\{1\}}$ | $0.1643^{\{7\}}$ | $0.16361^{\{6\}}$ | $0.15974^{\{2\}}$ | $0.16824^{\{9\}}$ | $0.16164^{\{4\}}$ | $0.1606^{\{3\}}$ | $0.16255^{\{5\}}$ | $0.16646^{\{8\}}$ |
| | | $\hat{\theta}$ | $0.07623^{\{2\}}$ | $0.07959^{\{8\}}$ | $0.07689^{\{3\}}$ | $0.07458^{\{1\}}$ | $0.08148^{\{9\}}$ | $0.07946^{\{6\}}$ | $0.07778^{\{4\}}$ | $0.07955^{\{7\}}$ | $0.07842^{\{5\}}$ |
| | MSE | $\hat{a}$ | $0.02357^{\{3\}}$ | $0.02392^{\{5\}}$ | $0.02171^{\{1\}}$ | $0.02484^{\{7\}}$ | $0.02491^{\{8\}}$ | $0.02466^{\{6\}}$ | $0.02273^{\{2\}}$ | $0.02371^{\{4\}}$ | $0.02576^{\{9\}}$ |
| | | $\hat{b}$ | $0.03709^{\{1\}}$ | $0.03955^{\{8\}}$ | $0.03878^{\{6\}}$ | $0.0377^{\{5\}}$ | $0.03988^{\{9\}}$ | $0.03767^{\{4\}}$ | $0.03712^{\{2\}}$ | $0.03732^{\{3\}}$ | $0.0393^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.00925^{\{2\}}$ | $0.01^{\{6\}}$ | $0.00941^{\{3\}}$ | $0.00902^{\{1\}}$ | $0.01033^{\{9\}}$ | $0.01008^{\{7\}}$ | $0.00972^{\{5\}}$ | $0.01009^{\{8\}}$ | $0.00957^{\{4\}}$ |
| | MRE | $\hat{a}$ | $0.07744^{\{3\}}$ | $0.07859^{\{6\}}$ | $0.07556^{\{1\}}$ | $0.07756^{\{4\}}$ | $0.0794^{\{7.5\}}$ | $0.0794^{\{7.5\}}$ | $0.07702^{\{2\}}$ | $0.07815^{\{5\}}$ | $0.07994^{\{9\}}$ |
| | | $\hat{b}$ | $0.31885^{\{1\}}$ | $0.32861^{\{7\}}$ | $0.32722^{\{6\}}$ | $0.31947^{\{2\}}$ | $0.33648^{\{9\}}$ | $0.32328^{\{4\}}$ | $0.32119^{\{3\}}$ | $0.3251^{\{5\}}$ | $0.33292^{\{8\}}$ |
| | | $\hat{\theta}$ | $0.30493^{\{2\}}$ | $0.31838^{\{8\}}$ | $0.30758^{\{3\}}$ | $0.29833^{\{1\}}$ | $0.32592^{\{9\}}$ | $0.31782^{\{6\}}$ | $0.3111^{\{4\}}$ | $0.31819^{\{7\}}$ | $0.31367^{\{5\}}$ |
| | $\sum Ranks$ | | $18^{\{1\}}$ | $61^{\{7\}}$ | $30^{\{4\}}$ | $27^{\{2.5\}}$ | $77.5^{\{9\}}$ | $51.5^{\{6\}}$ | $27^{\{2.5\}}$ | $49^{\{5\}}$ | $64^{\{8\}}$ |
| 600 | BIAS | $\hat{a}$ | $0.09438^{\{4\}}$ | $0.09615^{\{7\}}$ | $0.095^{\{5\}}$ | $0.08866^{\{1\}}$ | $0.09705^{\{8\}}$ | $0.09405^{\{3\}}$ | $0.09355^{\{2\}}$ | $0.09754^{\{9\}}$ | $0.0958^{\{6\}}$ |
| | | $\hat{b}$ | $0.15184^{\{1\}}$ | $0.1562^{\{6\}}$ | $0.15355^{\{3\}}$ | $0.15428^{\{4\}}$ | $0.15885^{\{8\}}$ | $0.15216^{\{2\}}$ | $0.1544^{\{5\}}$ | $0.15812^{\{7\}}$ | $0.16051^{\{9\}}$ |
| | | $\hat{\theta}$ | $0.07078^{\{1\}}$ | $0.07612^{\{9\}}$ | $0.07233^{\{4\}}$ | $0.07129^{\{2\}}$ | $0.0752^{\{7\}}$ | $0.07199^{\{3\}}$ | $0.07275^{\{5\}}$ | $0.07605^{\{8\}}$ | $0.07341^{\{6\}}$ |
| | MSE | $\hat{a}$ | $0.01509^{\{2\}}$ | $0.01557^{\{6\}}$ | $0.01549^{\{5\}}$ | $0.0135^{\{1\}}$ | $0.01701^{\{9\}}$ | $0.01524^{\{4\}}$ | $0.01523^{\{3\}}$ | $0.01605^{\{8\}}$ | $0.01579^{\{7\}}$ |
| | | $\hat{b}$ | $0.03426^{\{2\}}$ | $0.03556^{\{5\}}$ | $0.03505^{\{3\}}$ | $0.03529^{\{4\}}$ | $0.03649^{\{8\}}$ | $0.03397^{\{1\}}$ | $0.03565^{\{6\}}$ | $0.03623^{\{7\}}$ | $0.03727^{\{9\}}$ |
| | | $\hat{\theta}$ | $0.00834^{\{2\}}$ | $0.00929^{\{9\}}$ | $0.00855^{\{5\}}$ | $0.00814^{\{1\}}$ | $0.00899^{\{7\}}$ | $0.00835^{\{3\}}$ | $0.00846^{\{4\}}$ | $0.00912^{\{8\}}$ | $0.00859^{\{6\}}$ |
| | MRE | $\hat{a}$ | $0.06292^{\{4\}}$ | $0.0641^{\{7\}}$ | $0.06334^{\{5\}}$ | $0.05911^{\{1\}}$ | $0.0647^{\{8\}}$ | $0.0627^{\{3\}}$ | $0.06237^{\{2\}}$ | $0.06503^{\{9\}}$ | $0.06387^{\{6\}}$ |
| | | $\hat{b}$ | $0.30368^{\{1\}}$ | $0.31241^{\{6\}}$ | $0.3071^{\{3\}}$ | $0.30857^{\{4\}}$ | $0.31769^{\{8\}}$ | $0.30432^{\{2\}}$ | $0.3088^{\{5\}}$ | $0.31624^{\{7\}}$ | $0.32103^{\{9\}}$ |
| | | $\hat{\theta}$ | $0.28314^{\{1\}}$ | $0.30447^{\{9\}}$ | $0.28931^{\{4\}}$ | $0.28515^{\{2\}}$ | $0.3008^{\{7\}}$ | $0.28798^{\{3\}}$ | $0.29099^{\{5\}}$ | $0.3042^{\{8\}}$ | $0.29365^{\{6\}}$ |
| | $\sum Ranks$ | | $18^{\{1\}}$ | $64^{\{6.5\}}$ | $37^{\{4.5\}}$ | $20^{\{2\}}$ | $70^{\{8\}}$ | $24^{\{3\}}$ | $37^{\{4.5\}}$ | $71^{\{9\}}$ | $64^{\{6.5\}}$ |

Table 3.2: BIAS, MSE and MRE for (a=2.5, b=0.75, theta=1.5).

| n | Est. | Est. Par. | MLE | ADE | CVME | MPSE | LSE | RTADE | WLSE | MSADE | MSALDE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 35 | BIAS | $\hat{a}$ | $0.71836^{\{7\}}$ | $0.6894^{\{2.5\}}$ | $0.70916^{\{6\}}$ | $0.7015^{\{5\}}$ | $0.6894^{\{2.5\}}$ | $0.72325^{\{9\}}$ | $0.71929^{\{8\}}$ | $0.69872^{\{4\}}$ | $0.6817^{\{1\}}$ |
| | | $\hat{b}$ | $0.26312^{\{2\}}$ | $0.27062^{\{7\}}$ | $0.26981^{\{5\}}$ | $0.26689^{\{4\}}$ | $0.27005^{\{6\}}$ | $0.27093^{\{8\}}$ | $0.2759^{\{9\}}$ | $0.26641^{\{3\}}$ | $0.26285^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.54919^{\{4\}}$ | $0.57489^{\{9\}}$ | $0.56226^{\{7\}}$ | $0.55146^{\{6\}}$ | $0.53477^{\{2\}}$ | $0.57454^{\{8\}}$ | $0.55145^{\{5\}}$ | $0.53078^{\{1\}}$ | $0.53609^{\{3\}}$ |
| | MSE | $\hat{a}$ | $0.872^{\{7\}}$ | $0.7912^{\{2\}}$ | $0.8434^{\{6\}}$ | $0.83293^{\{5\}}$ | $0.7975^{\{3\}}$ | $0.90819^{\{9\}}$ | $0.87631^{\{8\}}$ | $0.83168^{\{4\}}$ | $0.78701^{\{1\}}$ |
| | | $\hat{b}$ | $0.09486^{\{2\}}$ | $0.09939^{\{7\}}$ | $0.09725^{\{3\}}$ | $0.09773^{\{4\}}$ | $0.09833^{\{5\}}$ | $0.1002^{\{8\}}$ | $0.10267^{\{9\}}$ | $0.0984^{\{6\}}$ | $0.09441^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.47229^{\{6\}}$ | $0.50846^{\{9\}}$ | $0.48132^{\{7\}}$ | $0.46675^{\{5\}}$ | $0.44206^{\{2\}}$ | $0.50497^{\{8\}}$ | $0.46201^{\{4\}}$ | $0.43668^{\{1\}}$ | $0.4464^{\{3\}}$ |
| | MRE | $\hat{a}$ | $0.28734^{\{7\}}$ | $0.27576^{\{2.5\}}$ | $0.28366^{\{6\}}$ | $0.2806^{\{5\}}$ | $0.27576^{\{2.5\}}$ | $0.2893^{\{9\}}$ | $0.28772^{\{8\}}$ | $0.27949^{\{4\}}$ | $0.27268^{\{1\}}$ |
| | | $\hat{b}$ | $0.35083^{\{2\}}$ | $0.36082^{\{7\}}$ | $0.35975^{\{5\}}$ | $0.35586^{\{4\}}$ | $0.36007^{\{6\}}$ | $0.36124^{\{8\}}$ | $0.36787^{\{9\}}$ | $0.35522^{\{3\}}$ | $0.35046^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.36613^{\{4\}}$ | $0.38326^{\{9\}}$ | $0.37484^{\{7\}}$ | $0.36764^{\{6\}}$ | $0.35651^{\{2\}}$ | $0.38302^{\{8\}}$ | $0.36763^{\{5\}}$ | $0.35385^{\{1\}}$ | $0.35739^{\{3\}}$ |
| | $\sum Ranks$ | | $41^{\{4\}}$ | $55^{\{7\}}$ | $52^{\{6\}}$ | $44^{\{5\}}$ | $31^{\{3\}}$ | $75^{\{9\}}$ | $65^{\{8\}}$ | $27^{\{2\}}$ | $15^{\{1\}}$ |
| 100 | BIAS | $\hat{a}$ | $0.47922^{\{4\}}$ | $0.47765^{\{2\}}$ | $0.48235^{\{6\}}$ | $0.48368^{\{7\}}$ | $0.47994^{\{5\}}$ | $0.49385^{\{8\}}$ | $0.47772^{\{3\}}$ | $0.50474^{\{9\}}$ | $0.47304^{\{1\}}$ |
| | | $\hat{b}$ | $0.26097^{\{4\}}$ | $0.26335^{\{6\}}$ | $0.26552^{\{8\}}$ | $0.26134^{\{5\}}$ | $0.25625^{\{2\}}$ | $0.26618^{\{9\}}$ | $0.26439^{\{7\}}$ | $0.25872^{\{3\}}$ | $0.25467^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.523^{\{6\}}$ | $0.53008^{\{7\}}$ | $0.51742^{\{3\}}$ | $0.51902^{\{4\}}$ | $0.50309^{\{1\}}$ | $0.53454^{\{8\}}$ | $0.53465^{\{9\}}$ | $0.51944^{\{5\}}$ | $0.50557^{\{2\}}$ |
| | MSE | $\hat{a}$ | $0.43547^{\{7\}}$ | $0.42411^{\{2\}}$ | $0.43255^{\{4\}}$ | $0.43503^{\{6\}}$ | $0.43377^{\{5\}}$ | $0.44956^{\{8\}}$ | $0.41801^{\{1\}}$ | $0.47164^{\{9\}}$ | $0.42616^{\{3\}}$ |
| | | $\hat{b}$ | $0.09225^{\{4\}}$ | $0.09466^{\{7\}}$ | $0.09665^{\{8\}}$ | $0.09419^{\{6\}}$ | $0.08997^{\{1\}}$ | $0.0972^{\{9\}}$ | $0.09316^{\{5\}}$ | $0.09043^{\{3\}}$ | $0.09019^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.42671^{\{6\}}$ | $0.4442^{\{8\}}$ | $0.41432^{\{3\}}$ | $0.42425^{\{5\}}$ | $0.40661^{\{2\}}$ | $0.44551^{\{9\}}$ | $0.44065^{\{7\}}$ | $0.41949^{\{4\}}$ | $0.39631^{\{1\}}$ |
| | MRE | $\hat{a}$ | $0.19169^{\{4\}}$ | $0.19106^{\{2\}}$ | $0.19294^{\{6\}}$ | $0.19347^{\{7\}}$ | $0.19198^{\{5\}}$ | $0.19754^{\{8\}}$ | $0.19109^{\{3\}}$ | $0.20189^{\{9\}}$ | $0.18922^{\{1\}}$ |
| | | $\hat{b}$ | $0.34796^{\{4\}}$ | $0.35114^{\{6\}}$ | $0.35403^{\{8\}}$ | $0.34845^{\{5\}}$ | $0.34166^{\{2\}}$ | $0.3549^{\{9\}}$ | $0.35252^{\{7\}}$ | $0.34497^{\{3\}}$ | $0.33956^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.34867^{\{6\}}$ | $0.35338^{\{7\}}$ | $0.34495^{\{3\}}$ | $0.34601^{\{4\}}$ | $0.33539^{\{1\}}$ | $0.35636^{\{8\}}$ | $0.35643^{\{9\}}$ | $0.34629^{\{5\}}$ | $0.33705^{\{2\}}$ |
| | $\sum Ranks$ | | $45^{\{3\}}$ | $47^{\{4\}}$ | $49^{\{5.5\}}$ | $49^{\{5.5\}}$ | $24^{\{2\}}$ | $76^{\{9\}}$ | $51^{\{8\}}$ | $50^{\{7\}}$ | $14^{\{1\}}$ |
| 250 | BIAS | $\hat{a}$ | $0.331^{\{9\}}$ | $0.30906^{\{3\}}$ | $0.31852^{\{6\}}$ | $0.308^{\{2\}}$ | $0.31277^{\{4\}}$ | $0.32367^{\{8\}}$ | $0.30532^{\{1\}}$ | $0.32252^{\{7\}}$ | $0.31452^{\{5\}}$ |
| | | $\hat{b}$ | $0.24507^{\{6\}}$ | $0.24212^{\{2\}}$ | $0.24959^{\{9\}}$ | $0.24379^{\{5\}}$ | $0.23689^{\{1\}}$ | $0.24924^{\{7.5\}}$ | $0.24289^{\{3\}}$ | $0.24344^{\{4\}}$ | $0.24924^{\{7.5\}}$ |
| | | $\hat{\theta}$ | $0.45478^{\{5\}}$ | $0.44322^{\{1\}}$ | $0.45954^{\{8\}}$ | $0.447^{\{4\}}$ | $0.44519^{\{2\}}$ | $0.45905^{\{7\}}$ | $0.44583^{\{3\}}$ | $0.45604^{\{6\}}$ | $0.46229^{\{9\}}$ |
| | MSE | $\hat{a}$ | $0.20458^{\{9\}}$ | $0.18174^{\{5\}}$ | $0.1892^{\{6\}}$ | $0.18108^{\{4\}}$ | $0.17948^{\{3\}}$ | $0.2011^{\{8\}}$ | $0.17259^{\{1\}}$ | $0.19118^{\{7\}}$ | $0.17752^{\{2\}}$ |
| | | $\hat{b}$ | $0.08736^{\{7\}}$ | $0.08369^{\{3\}}$ | $0.08801^{\{8\}}$ | $0.0853^{\{5\}}$ | $0.07804^{\{1\}}$ | $0.08608^{\{6\}}$ | $0.08264^{\{2\}}$ | $0.08485^{\{4\}}$ | $0.08856^{\{9\}}$ |
| | | $\hat{\theta}$ | $0.33734^{\{8\}}$ | $0.31381^{\{1\}}$ | $0.33359^{\{5\}}$ | $0.3242^{\{4\}}$ | $0.31773^{\{2\}}$ | $0.33605^{\{7\}}$ | $0.32157^{\{3\}}$ | $0.33386^{\{6\}}$ | $0.34121^{\{9\}}$ |
| | MRE | $\hat{a}$ | $0.1324^{\{9\}}$ | $0.12363^{\{3\}}$ | $0.12741^{\{6\}}$ | $0.1232^{\{2\}}$ | $0.12511^{\{4\}}$ | $0.12947^{\{8\}}$ | $0.12213^{\{1\}}$ | $0.12901^{\{7\}}$ | $0.12581^{\{5\}}$ |
| | | $\hat{b}$ | $0.32675^{\{6\}}$ | $0.32283^{\{2\}}$ | $0.33279^{\{9\}}$ | $0.32506^{\{5\}}$ | $0.31585^{\{1\}}$ | $0.33231^{\{7\}}$ | $0.32386^{\{3\}}$ | $0.32459^{\{4\}}$ | $0.33232^{\{8\}}$ |
| | | $\hat{\theta}$ | $0.30318^{\{5\}}$ | $0.29548^{\{1\}}$ | $0.30636^{\{8\}}$ | $0.298^{\{4\}}$ | $0.29679^{\{2\}}$ | $0.30604^{\{7\}}$ | $0.29722^{\{3\}}$ | $0.30403^{\{6\}}$ | $0.3082^{\{9\}}$ |
| | $\sum Ranks$ | | $64^{\{7\}}$ | $21^{\{3\}}$ | $65^{\{8\}}$ | $35^{\{4\}}$ | $20^{\{1.5\}}$ | $65.5^{\{9\}}$ | $20^{\{1.5\}}$ | $51^{\{5\}}$ | $63.5^{\{6\}}$ |
| 400 | BIAS | $\hat{a}$ | $0.2318^{\{1\}}$ | $0.24528^{\{5\}}$ | $0.24575^{\{7\}}$ | $0.24802^{\{8\}}$ | $0.23968^{\{3\}}$ | $0.24926^{\{9\}}$ | $0.2385^{\{2\}}$ | $0.24549^{\{6\}}$ | $0.24525^{\{4\}}$ |
| | | $\hat{b}$ | $0.224^{\{4\}}$ | $0.23807^{\{8\}}$ | $0.22398^{\{3\}}$ | $0.23248^{\{7\}}$ | $0.22162^{\{1\}}$ | $0.23842^{\{9\}}$ | $0.2232^{\{2\}}$ | $0.23112^{\{6\}}$ | $0.22725^{\{5\}}$ |
| | | $\hat{\theta}$ | $0.38425^{\{5\}}$ | $0.40013^{\{8\}}$ | $0.37585^{\{3\}}$ | $0.3942^{\{7\}}$ | $0.3738^{\{2\}}$ | $0.41042^{\{9\}}$ | $0.37253^{\{1\}}$ | $0.38983^{\{6\}}$ | $0.37917^{\{4\}}$ |
| | MSE | $\hat{a}$ | $0.10112^{\{1\}}$ | $0.1089^{\{5\}}$ | $0.11022^{\{6\}}$ | $0.10761^{\{4\}}$ | $0.10718^{\{3\}}$ | $0.11384^{\{8\}}$ | $0.10652^{\{2\}}$ | $0.11144^{\{7\}}$ | $0.11483^{\{9\}}$ |
| | | $\hat{b}$ | $0.07488^{\{2\}}$ | $0.08355^{\{9\}}$ | $0.07595^{\{4\}}$ | $0.07915^{\{6\}}$ | $0.07523^{\{3\}}$ | $0.08237^{\{8\}}$ | $0.07401^{\{1\}}$ | $0.07965^{\{7\}}$ | $0.07785^{\{5\}}$ |
| | | $\hat{\theta}$ | $0.2419^{\{5\}}$ | $0.2548^{\{8\}}$ | $0.23013^{\{2\}}$ | $0.24619^{\{6\}}$ | $0.23484^{\{3\}}$ | $0.26794^{\{9\}}$ | $0.22588^{\{1\}}$ | $0.24709^{\{7\}}$ | $0.23965^{\{4\}}$ |
| | MRE | $\hat{a}$ | $0.09272^{\{1\}}$ | $0.09811^{\{5\}}$ | $0.0983^{\{7\}}$ | $0.09921^{\{8\}}$ | $0.09587^{\{3\}}$ | $0.09971^{\{9\}}$ | $0.0954^{\{2\}}$ | $0.0982^{\{6\}}$ | $0.0981^{\{4\}}$ |
| | | $\hat{b}$ | $0.29867^{\{4\}}$ | $0.31742^{\{8\}}$ | $0.29864^{\{3\}}$ | $0.30998^{\{7\}}$ | $0.29549^{\{1\}}$ | $0.3179^{\{9\}}$ | $0.2976^{\{2\}}$ | $0.30816^{\{6\}}$ | $0.303^{\{5\}}$ |
| | | $\hat{\theta}$ | $0.25617^{\{5\}}$ | $0.26675^{\{8\}}$ | $0.25057^{\{3\}}$ | $0.2628^{\{7\}}$ | $0.2492^{\{2\}}$ | $0.27362^{\{9\}}$ | $0.24835^{\{1\}}$ | $0.25989^{\{6\}}$ | $0.25278^{\{4\}}$ |
| | $\sum Ranks$ | | $28^{\{3\}}$ | $64^{\{8\}}$ | $38^{\{4\}}$ | $60^{\{7\}}$ | $21^{\{2\}}$ | $79^{\{9\}}$ | $14^{\{1\}}$ | $57^{\{6\}}$ | $44^{\{5\}}$ |
| 600 | BIAS | $\hat{a}$ | $0.19689^{\{4\}}$ | $0.19976^{\{5\}}$ | $0.19339^{\{2\}}$ | $0.20318^{\{8\}}$ | $0.20538^{\{9\}}$ | $0.19365^{\{3\}}$ | $0.20026^{\{6\}}$ | $0.18748^{\{1\}}$ | $0.20144^{\{7\}}$ |
| | | $\hat{b}$ | $0.21033^{\{4\}}$ | $0.21286^{\{8\}}$ | $0.20614^{\{2\}}$ | $0.21246^{\{7\}}$ | $0.21493^{\{9\}}$ | $0.20999^{\{3\}}$ | $0.20072^{\{1\}}$ | $0.21109^{\{5\}}$ | $0.21229^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.33473^{\{6\}}$ | $0.33223^{\{5\}}$ | $0.3308^{\{4\}}$ | $0.34344^{\{8\}}$ | $0.355^{\{9\}}$ | $0.32483^{\{2\}}$ | $0.31764^{\{1\}}$ | $0.32914^{\{3\}}$ | $0.33812^{\{7\}}$ |
| | MSE | $\hat{a}$ | $0.07173^{\{5\}}$ | $0.07335^{\{6\}}$ | $0.07017^{\{4\}}$ | $0.07688^{\{8\}}$ | $0.07878^{\{9\}}$ | $0.06657^{\{2\}}$ | $0.06937^{\{3\}}$ | $0.06402^{\{1\}}$ | $0.07595^{\{7\}}$ |
| | | $\hat{b}$ | $0.0689^{\{5\}}$ | $0.06856^{\{4\}}$ | $0.06588^{\{2\}}$ | $0.07092^{\{9\}}$ | $0.07075^{\{8\}}$ | $0.0684^{\{3\}}$ | $0.06245^{\{1\}}$ | $0.06953^{\{6\}}$ | $0.06972^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.18464^{\{6\}}$ | $0.17392^{\{3\}}$ | $0.18068^{\{5\}}$ | $0.19215^{\{8\}}$ | $0.2042^{\{9\}}$ | $0.16652^{\{2\}}$ | $0.16386^{\{1\}}$ | $0.17799^{\{4\}}$ | $0.187^{\{7\}}$ |
| | MRE | $\hat{a}$ | $0.07876^{\{4\}}$ | $0.0799^{\{5\}}$ | $0.07736^{\{2\}}$ | $0.08127^{\{8\}}$ | $0.08215^{\{9\}}$ | $0.07746^{\{3\}}$ | $0.0801^{\{6\}}$ | $0.07499^{\{1\}}$ | $0.08058^{\{7\}}$ |
| | | $\hat{b}$ | $0.28044^{\{4\}}$ | $0.28382^{\{8\}}$ | $0.27486^{\{2\}}$ | $0.28328^{\{7\}}$ | $0.28657^{\{9\}}$ | $0.27999^{\{3\}}$ | $0.26763^{\{1\}}$ | $0.28145^{\{5\}}$ | $0.28306^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.22315^{\{6\}}$ | $0.22149^{\{5\}}$ | $0.22053^{\{4\}}$ | $0.22896^{\{8\}}$ | $0.23666^{\{9\}}$ | $0.21655^{\{2\}}$ | $0.21176^{\{1\}}$ | $0.21943^{\{3\}}$ | $0.22541^{\{7\}}$ |
| | $\sum Ranks$ | | $44^{\{5\}}$ | $49^{\{6\}}$ | $27^{\{3\}}$ | $71^{\{8\}}$ | $80^{\{9\}}$ | $23^{\{2\}}$ | $21^{\{1\}}$ | $29^{\{4\}}$ | $61^{\{7\}}$ |

Table 3.3: BIAS, MSE and MRE for (a=2.0, b=1.5, theta=0.5)

| n | Est. | Est. Par. | MLE | ADE | CVME | MPSE | LSE | RTADE | WLSE | MSADE | MSALDE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 35 | BIAS | $\hat{a}$ | $0.40401^{\{3\}}$ | $0.41613^{\{8\}}$ | $0.40746^{\{4\}}$ | $0.42101^{\{9\}}$ | $0.41133^{\{6\}}$ | $0.4095^{\{5\}}$ | $0.41215^{\{7\}}$ | $0.38922^{\{1\}}$ | $0.40114^{\{2\}}$ |
| | | $\hat{b}$ | $0.65918^{\{8\}}$ | $0.63204^{\{4\}}$ | $0.62999^{\{3\}}$ | $0.6424^{\{6\}}$ | $0.6269^{\{1\}}$ | $0.67124^{\{9\}}$ | $0.6397^{\{5\}}$ | $0.64671^{\{7\}}$ | $0.62763^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.19084^{\{6\}}$ | $0.18732^{\{2\}}$ | $0.18861^{\{4\}}$ | $0.19464^{\{9\}}$ | $0.19329^{\{8\}}$ | $0.19308^{\{7\}}$ | $0.19066^{\{5\}}$ | $0.18753^{\{3\}}$ | $0.18608^{\{1\}}$ |
| | MSE | $\hat{a}$ | $0.29097^{\{7\}}$ | $0.28955^{\{6\}}$ | $0.28345^{\{4\}}$ | $0.294^{\{8\}}$ | $0.28909^{\{5\}}$ | $0.2823^{\{3\}}$ | $0.29702^{\{9\}}$ | $0.25057^{\{1\}}$ | $0.27034^{\{2\}}$ |
| | | $\hat{b}$ | $0.60432^{\{8\}}$ | $0.55715^{\{4\}}$ | $0.55075^{\{3\}}$ | $0.56722^{\{6\}}$ | $0.54999^{\{2\}}$ | $0.61616^{\{9\}}$ | $0.56437^{\{5\}}$ | $0.57933^{\{7\}}$ | $0.54763^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.0528^{\{5\}}$ | $0.05124^{\{3\}}$ | $0.05153^{\{4\}}$ | $0.05463^{\{9\}}$ | $0.05433^{\{8\}}$ | $0.05285^{\{6\}}$ | $0.0531^{\{7\}}$ | $0.05011^{\{1\}}$ | $0.05025^{\{2\}}$ |
| | MRE | $\hat{a}$ | $0.20201^{\{3\}}$ | $0.20807^{\{8\}}$ | $0.20373^{\{4\}}$ | $0.21051^{\{9\}}$ | $0.20567^{\{6\}}$ | $0.20475^{\{5\}}$ | $0.20608^{\{7\}}$ | $0.19461^{\{1\}}$ | $0.20057^{\{2\}}$ |
| | | $\hat{b}$ | $0.43945^{\{8\}}$ | $0.42136^{\{4\}}$ | $0.42^{\{3\}}$ | $0.42826^{\{6\}}$ | $0.41794^{\{1\}}$ | $0.44749^{\{9\}}$ | $0.42647^{\{5\}}$ | $0.43114^{\{7\}}$ | $0.41842^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.38169^{\{6\}}$ | $0.37464^{\{2\}}$ | $0.37722^{\{4\}}$ | $0.38928^{\{9\}}$ | $0.38657^{\{8\}}$ | $0.38616^{\{7\}}$ | $0.38132^{\{5\}}$ | $0.37506^{\{3\}}$ | $0.37215^{\{1\}}$ |
| | $\sum Ranks$ | | $54^{\{6\}}$ | $41^{\{4\}}$ | $33^{\{3\}}$ | $71^{\{9\}}$ | $45^{\{5\}}$ | $60^{\{8\}}$ | $55^{\{7\}}$ | $31^{\{2\}}$ | $15^{\{1\}}$ |
| 100 | BIAS | $\hat{a}$ | $0.25095^{\{6\}}$ | $0.25828^{\{8\}}$ | $0.24918^{\{5\}}$ | $0.2521^{\{7\}}$ | $0.246^{\{3\}}$ | $0.24686^{\{4\}}$ | $0.26275^{\{9\}}$ | $0.24382^{\{1\}}$ | $0.24454^{\{2\}}$ |
| | | $\hat{b}$ | $0.60033^{\{9\}}$ | $0.57941^{\{1\}}$ | $0.58165^{\{2\}}$ | $0.5861^{\{3\}}$ | $0.59578^{\{6\}}$ | $0.58784^{\{5\}}$ | $0.59725^{\{7\}}$ | $0.5998^{\{8\}}$ | $0.58759^{\{4\}}$ |
| | | $\hat{\theta}$ | $0.18372^{\{6\}}$ | $0.18057^{\{3\}}$ | $0.1794^{\{1\}}$ | $0.18832^{\{9\}}$ | $0.18821^{\{8\}}$ | $0.18384^{\{7\}}$ | $0.18342^{\{5\}}$ | $0.17991^{\{2\}}$ | $0.18088^{\{4\}}$ |
| | MSE | $\hat{a}$ | $0.10854^{\{6\}}$ | $0.11429^{\{8\}}$ | $0.10736^{\{5\}}$ | $0.11076^{\{7\}}$ | $0.10101^{\{1\}}$ | $0.10376^{\{3\}}$ | $0.11643^{\{9\}}$ | $0.10298^{\{2\}}$ | $0.10398^{\{4\}}$ |
| | | $\hat{b}$ | $0.49408^{\{7\}}$ | $0.47296^{\{2\}}$ | $0.47462^{\{3\}}$ | $0.47488^{\{4\}}$ | $0.48069^{\{5\}}$ | $0.48248^{\{6\}}$ | $0.50212^{\{9\}}$ | $0.50111^{\{8\}}$ | $0.47245^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.05074^{\{5\}}$ | $0.05^{\{4\}}$ | $0.04976^{\{3\}}$ | $0.05346^{\{9\}}$ | $0.05228^{\{8\}}$ | $0.05127^{\{7\}}$ | $0.05085^{\{6\}}$ | $0.04861^{\{1\}}$ | $0.04914^{\{2\}}$ |
| | MRE | $\hat{a}$ | $0.12548^{\{6\}}$ | $0.12914^{\{8\}}$ | $0.12459^{\{5\}}$ | $0.12605^{\{7\}}$ | $0.123^{\{3\}}$ | $0.12343^{\{4\}}$ | $0.13138^{\{9\}}$ | $0.12191^{\{1\}}$ | $0.12227^{\{2\}}$ |
| | | $\hat{b}$ | $0.40022^{\{9\}}$ | $0.38627^{\{1\}}$ | $0.38777^{\{2\}}$ | $0.39073^{\{3\}}$ | $0.39719^{\{6\}}$ | $0.39189^{\{5\}}$ | $0.39817^{\{7\}}$ | $0.39986^{\{8\}}$ | $0.39172^{\{4\}}$ |
| | | $\hat{\theta}$ | $0.36744^{\{6\}}$ | $0.36113^{\{3\}}$ | $0.3588^{\{1\}}$ | $0.37664^{\{9\}}$ | $0.37643^{\{8\}}$ | $0.36769^{\{7\}}$ | $0.36683^{\{5\}}$ | $0.35981^{\{2\}}$ | $0.36177^{\{4\}}$ |
| | $\sum Ranks$ | | $60^{\{8\}}$ | $38^{\{4\}}$ | $27^{\{1.5\}}$ | $58^{\{7\}}$ | $48^{\{5.5\}}$ | $48^{\{5.5\}}$ | $66^{\{9\}}$ | $33^{\{3\}}$ | $27^{\{1.5\}}$ |
| 250 | BIAS | $\hat{a}$ | $0.15273^{\{2\}}$ | $0.16105^{\{8\}}$ | $0.15399^{\{5\}}$ | $0.15523^{\{6\}}$ | $0.15792^{\{7\}}$ | $0.16365^{\{9\}}$ | $0.15279^{\{3\}}$ | $0.14782^{\{1\}}$ | $0.15376^{\{4\}}$ |
| | | $\hat{b}$ | $0.5915^{\{8\}}$ | $0.56567^{\{1\}}$ | $0.57433^{\{4\}}$ | $0.57525^{\{5\}}$ | $0.57335^{\{2\}}$ | $0.58608^{\{7\}}$ | $0.59646^{\{9\}}$ | $0.57361^{\{3\}}$ | $0.58507^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.18086^{\{6\}}$ | $0.17935^{\{5\}}$ | $0.17854^{\{3\}}$ | $0.17898^{\{4\}}$ | $0.18404^{\{9\}}$ | $0.18146^{\{7\}}$ | $0.18311^{\{8\}}$ | $0.17773^{\{2\}}$ | $0.17734^{\{1\}}$ |
| | MSE | $\hat{a}$ | $0.03914^{\{3\}}$ | $0.04329^{\{8\}}$ | $0.03964^{\{4\}}$ | $0.04114^{\{6\}}$ | $0.04135^{\{7\}}$ | $0.04485^{\{9\}}$ | $0.03781^{\{2\}}$ | $0.03601^{\{1\}}$ | $0.03991^{\{5\}}$ |
| | | $\hat{b}$ | $0.48555^{\{9\}}$ | $0.44768^{\{1\}}$ | $0.46063^{\{5\}}$ | $0.45917^{\{4\}}$ | $0.44807^{\{2\}}$ | $0.46848^{\{6\}}$ | $0.48047^{\{8\}}$ | $0.45126^{\{3\}}$ | $0.47114^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.04999^{\{7\}}$ | $0.04914^{\{5\}}$ | $0.04912^{\{4\}}$ | $0.0486^{\{3\}}$ | $0.0517^{\{9\}}$ | $0.04956^{\{6\}}$ | $0.05014^{\{8\}}$ | $0.04832^{\{2\}}$ | $0.04763^{\{1\}}$ |
| | MRE | $\hat{a}$ | $0.07637^{\{2\}}$ | $0.08053^{\{8\}}$ | $0.077^{\{5\}}$ | $0.07761^{\{6\}}$ | $0.07896^{\{7\}}$ | $0.08183^{\{9\}}$ | $0.07639^{\{3\}}$ | $0.07391^{\{1\}}$ | $0.07688^{\{4\}}$ |
| | | $\hat{b}$ | $0.39433^{\{8\}}$ | $0.37711^{\{1\}}$ | $0.38289^{\{4\}}$ | $0.3835^{\{5\}}$ | $0.38223^{\{2\}}$ | $0.39072^{\{7\}}$ | $0.39764^{\{9\}}$ | $0.3824^{\{3\}}$ | $0.39005^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.36172^{\{6\}}$ | $0.35871^{\{5\}}$ | $0.35709^{\{3\}}$ | $0.35797^{\{4\}}$ | $0.36807^{\{9\}}$ | $0.36293^{\{7\}}$ | $0.36622^{\{8\}}$ | $0.35546^{\{2\}}$ | $0.35468^{\{1\}}$ |
| | $\sum Ranks$ | | $51^{\{6\}}$ | $42^{\{4\}}$ | $37^{\{3\}}$ | $43^{\{5\}}$ | $54^{\{7\}}$ | $67^{\{9\}}$ | $58^{\{8\}}$ | $18^{\{1\}}$ | $35^{\{2\}}$ |
| 400 | BIAS | $\hat{a}$ | $0.12418^{\{3\}}$ | $0.12428^{\{4\}}$ | $0.13181^{\{9\}}$ | $0.12607^{\{7\}}$ | $0.12603^{\{6\}}$ | $0.1249^{\{5\}}$ | $0.12286^{\{2\}}$ | $0.11795^{\{1\}}$ | $0.13113^{\{8\}}$ |
| | | $\hat{b}$ | $0.55058^{\{4\}}$ | $0.54638^{\{1\}}$ | $0.54999^{\{3\}}$ | $0.5624^{\{8\}}$ | $0.56793^{\{9\}}$ | $0.54835^{\{2\}}$ | $0.55737^{\{7\}}$ | $0.55088^{\{5\}}$ | $0.5565^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.17159^{\{1\}}$ | $0.172^{\{2\}}$ | $0.17247^{\{3\}}$ | $0.17767^{\{8\}}$ | $0.17932^{\{9\}}$ | $0.17362^{\{6\}}$ | $0.17301^{\{4\}}$ | $0.17611^{\{7\}}$ | $0.17317^{\{5\}}$ |
| | MSE | $\hat{a}$ | $0.02463^{\{2\}}$ | $0.02572^{\{4\}}$ | $0.02864^{\{8.5\}}$ | $0.02578^{\{5\}}$ | $0.02714^{\{7\}}$ | $0.02603^{\{6\}}$ | $0.02473^{\{3\}}$ | $0.02355^{\{1\}}$ | $0.02864^{\{8.5\}}$ |
| | | $\hat{b}$ | $0.41886^{\{3\}}$ | $0.41329^{\{1\}}$ | $0.42355^{\{5\}}$ | $0.43577^{\{8\}}$ | $0.44131^{\{9\}}$ | $0.42103^{\{4\}}$ | $0.42946^{\{6\}}$ | $0.41516^{\{2\}}$ | $0.43299^{\{7\}}$ |
| | | $\hat{\theta}$ | $0.04454^{\{1\}}$ | $0.04587^{\{3\}}$ | $0.04641^{\{5\}}$ | $0.04777^{\{8\}}$ | $0.04908^{\{9\}}$ | $0.04739^{\{6\}}$ | $0.04597^{\{4\}}$ | $0.04751^{\{7\}}$ | $0.04583^{\{2\}}$ |
| | MRE | $\hat{a}$ | $0.06209^{\{3\}}$ | $0.06214^{\{4\}}$ | $0.0659^{\{9\}}$ | $0.06303^{\{7\}}$ | $0.06301^{\{6\}}$ | $0.06245^{\{5\}}$ | $0.06143^{\{2\}}$ | $0.05898^{\{1\}}$ | $0.06557^{\{8\}}$ |
| | | $\hat{b}$ | $0.36705^{\{4\}}$ | $0.36426^{\{1\}}$ | $0.36666^{\{3\}}$ | $0.37493^{\{8\}}$ | $0.37862^{\{9\}}$ | $0.36557^{\{2\}}$ | $0.37158^{\{7\}}$ | $0.36725^{\{5\}}$ | $0.371^{\{6\}}$ |
| | | $\hat{\theta}$ | $0.34319^{\{1\}}$ | $0.34399^{\{2\}}$ | $0.34493^{\{3\}}$ | $0.35534^{\{8\}}$ | $0.35865^{\{9\}}$ | $0.34724^{\{6\}}$ | $0.34603^{\{4\}}$ | $0.35223^{\{7\}}$ | $0.34634^{\{5\}}$ |
| | $\sum Ranks$ | | $22^{\{1.5\}}$ | $22^{\{1.5\}}$ | $48.5^{\{6\}}$ | $67^{\{8\}}$ | $73^{\{9\}}$ | $42^{\{5\}}$ | $39^{\{4\}}$ | $36^{\{3\}}$ | $55.5^{\{7\}}$ |
| 600 | BIAS | $\hat{a}$ | $0.10053^{\{2\}}$ | $0.10264^{\{5\}}$ | $0.10334^{\{6\}}$ | $0.10019^{\{1\}}$ | $0.10076^{\{4\}}$ | $0.10418^{\{8\}}$ | $0.10055^{\{3\}}$ | $0.10592^{\{9\}}$ | $0.10414^{\{7\}}$ |
| | | $\hat{b}$ | $0.5424^{\{8\}}$ | $0.52844^{\{3\}}$ | $0.53022^{\{4\}}$ | $0.5372^{\{6\}}$ | $0.53479^{\{5\}}$ | $0.5236^{\{2\}}$ | $0.55149^{\{9\}}$ | $0.5413^{\{7\}}$ | $0.52248^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.16705^{\{7\}}$ | $0.16212^{\{3\}}$ | $0.16899^{\{8\}}$ | $0.16357^{\{5\}}$ | $0.1639^{\{6\}}$ | $0.15908^{\{1\}}$ | $0.17071^{\{9\}}$ | $0.16339^{\{4\}}$ | $0.16174^{\{2\}}$ |
| | MSE | $\hat{a}$ | $0.0169^{\{4\}}$ | $0.01693^{\{5\}}$ | $0.0176^{\{6\}}$ | $0.01682^{\{3\}}$ | $0.01664^{\{2\}}$ | $0.01766^{\{7\}}$ | $0.01657^{\{1\}}$ | $0.01808^{\{9\}}$ | $0.01789^{\{8\}}$ |
| | | $\hat{b}$ | $0.41807^{\{8\}}$ | $0.39602^{\{4\}}$ | $0.39226^{\{2\}}$ | $0.40775^{\{6\}}$ | $0.40041^{\{5\}}$ | $0.39476^{\{3\}}$ | $0.42006^{\{9\}}$ | $0.40783^{\{7\}}$ | $0.38721^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.04359^{\{7\}}$ | $0.04127^{\{3\}}$ | $0.04424^{\{8\}}$ | $0.04155^{\{4\}}$ | $0.04177^{\{6\}}$ | $0.03953^{\{1\}}$ | $0.04448^{\{9\}}$ | $0.04163^{\{5\}}$ | $0.04121^{\{2\}}$ |
| | MRE | $\hat{a}$ | $0.05026^{\{2\}}$ | $0.05132^{\{5\}}$ | $0.05167^{\{6\}}$ | $0.0501^{\{1\}}$ | $0.05038^{\{4\}}$ | $0.05209^{\{8\}}$ | $0.05028^{\{3\}}$ | $0.05296^{\{9\}}$ | $0.05207^{\{7\}}$ |
| | | $\hat{b}$ | $0.3616^{\{8\}}$ | $0.35229^{\{3\}}$ | $0.35348^{\{4\}}$ | $0.35813^{\{6\}}$ | $0.35653^{\{5\}}$ | $0.34906^{\{2\}}$ | $0.36766^{\{9\}}$ | $0.36087^{\{7\}}$ | $0.34832^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.33409^{\{7\}}$ | $0.32424^{\{3\}}$ | $0.33798^{\{8\}}$ | $0.32713^{\{5\}}$ | $0.3278^{\{6\}}$ | $0.31816^{\{1\}}$ | $0.34141^{\{9\}}$ | $0.32678^{\{4\}}$ | $0.32348^{\{2\}}$ |
| | $\sum Ranks$ | | $53^{\{7\}}$ | $34^{\{3\}}$ | $52^{\{6\}}$ | $37^{\{4\}}$ | $43^{\{5\}}$ | $33^{\{2\}}$ | $61^{\{8.5\}}$ | $61^{\{8.5\}}$ | $31^{\{1\}}$ |

Table 3.4: BIAS, MSE and MRE for (a=3.5, b=0.2, theta=0.9)

| n | Est. | Est. Par. | MLE | ADE | CVME | MPSE | LSE | RTADE | WLSE | MSADE | MSALDE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 35 | BIAS | $\hat{a}$ | $1.11231^{\{7\}}$ | $1.08021^{\{2\}}$ | $1.0966^{\{3\}}$ | $1.1179^{\{8\}}$ | $1.11148^{\{5\}}$ | $1.11157^{\{6\}}$ | $1.14508^{\{9\}}$ | $1.02864^{\{1\}}$ | $1.10167^{\{4\}}$ |
| | | $\hat{b}$ | $0.0706^{\{8\}}$ | $0.06965^{\{4\}}$ | $0.06985^{\{5.5\}}$ | $0.06985^{\{5.5\}}$ | $0.06788^{\{1\}}$ | $0.07205^{\{9\}}$ | $0.06942^{\{3\}}$ | $0.07013^{\{7\}}$ | $0.06839^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.23338^{\{4\}}$ | $0.24155^{\{7\}}$ | $0.23727^{\{5\}}$ | $0.24464^{\{8\}}$ | $0.22909^{\{2\}}$ | $0.24518^{\{9\}}$ | $0.22664^{\{1\}}$ | $0.23854^{\{6\}}$ | $0.22994^{\{3\}}$ |
| | MSE | $\hat{a}$ | $1.97278^{\{7\}}$ | $1.86501^{\{2\}}$ | $1.92368^{\{5\}}$ | $1.96395^{\{6\}}$ | $1.92296^{\{4\}}$ | $1.99038^{\{8\}}$ | $2.02977^{\{9\}}$ | $1.74002^{\{1\}}$ | $1.87597^{\{3\}}$ |
| | | $\hat{b}$ | $0.0075^{\{8\}}$ | $0.00738^{\{5\}}$ | $0.00723^{\{3\}}$ | $0.00737^{\{4\}}$ | $0.00702^{\{1\}}$ | $0.00771^{\{9\}}$ | $0.0074^{\{6\}}$ | $0.00748^{\{7\}}$ | $0.00711^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.08699^{\{4\}}$ | $0.09258^{\{7\}}$ | $0.08516^{\{3\}}$ | $0.0944^{\{9\}}$ | $0.08447^{\{2\}}$ | $0.09312^{\{8\}}$ | $0.08228^{\{1\}}$ | $0.09249^{\{6\}}$ | $0.08719^{\{5\}}$ |
| | MRE | $\hat{a}$ | $0.3178^{\{7\}}$ | $0.30863^{\{2\}}$ | $0.31331^{\{3\}}$ | $0.3194^{\{8\}}$ | $0.31757^{\{5\}}$ | $0.31759^{\{6\}}$ | $0.32717^{\{9\}}$ | $0.2939^{\{1\}}$ | $0.31476^{\{4\}}$ |
| | | $\hat{b}$ | $0.353^{\{8\}}$ | $0.34825^{\{4\}}$ | $0.34924^{\{5.5\}}$ | $0.34924^{\{5.5\}}$ | $0.33942^{\{1\}}$ | $0.36025^{\{9\}}$ | $0.34709^{\{3\}}$ | $0.35066^{\{7\}}$ | $0.34194^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.25931^{\{4\}}$ | $0.26839^{\{7\}}$ | $0.26363^{\{5\}}$ | $0.27182^{\{8\}}$ | $0.25455^{\{2\}}$ | $0.27242^{\{9\}}$ | $0.25182^{\{1\}}$ | $0.26504^{\{6\}}$ | $0.25549^{\{3\}}$ |
| | $\sum Ranks$ | | $57^{\{7\}}$ | $40^{\{4\}}$ | $38^{\{3\}}$ | $62^{\{8\}}$ | $23^{\{1\}}$ | $73^{\{9\}}$ | $42^{\{5.5\}}$ | $42^{\{5.5\}}$ | $28^{\{2\}}$ |
| 100 | BIAS | $\hat{a}$ | $0.95109^{\{3\}}$ | $0.98818^{\{9\}}$ | $0.95322^{\{4\}}$ | $0.94664^{\{2\}}$ | $0.97491^{\{5\}}$ | $0.9226^{\{1\}}$ | $0.98181^{\{6\}}$ | $0.98367^{\{7\}}$ | $0.98633^{\{8\}}$ |
| | | $\hat{b}$ | $0.06252^{\{5\}}$ | $0.063^{\{7\}}$ | $0.066^{\{9\}}$ | $0.06383^{\{8\}}$ | $0.06146^{\{2\}}$ | $0.06275^{\{6\}}$ | $0.06215^{\{3\}}$ | $0.06081^{\{1\}}$ | $0.06231^{\{4\}}$ |
| | | $\hat{\theta}$ | $0.22458^{\{3\}}$ | $0.23845^{\{9\}}$ | $0.23456^{\{8\}}$ | $0.22529^{\{5\}}$ | $0.226^{\{6\}}$ | $0.22296^{\{1\}}$ | $0.22498^{\{4\}}$ | $0.22868^{\{7\}}$ | $0.22409^{\{2\}}$ |
| | MSE | $\hat{a}$ | $1.57368^{\{4\}}$ | $1.6596^{\{9\}}$ | $1.54491^{\{3\}}$ | $1.52937^{\{2\}}$ | $1.60339^{\{5\}}$ | $1.44943^{\{1\}}$ | $1.64625^{\{8\}}$ | $1.63772^{\{7\}}$ | $1.61158^{\{6\}}$ |
| | | $\hat{b}$ | $0.00604^{\{4.5\}}$ | $0.00605^{\{6\}}$ | $0.0067^{\{9\}}$ | $0.00632^{\{8\}}$ | $0.00596^{\{2\}}$ | $0.00611^{\{7\}}$ | $0.00601^{\{3\}}$ | $0.0057^{\{1\}}$ | $0.00604^{\{4.5\}}$ |
| | | $\hat{\theta}$ | $0.07959^{\{5\}}$ | $0.08894^{\{9\}}$ | $0.08302^{\{6\}}$ | $0.07743^{\{1\}}$ | $0.08395^{\{7\}}$ | $0.079^{\{3\}}$ | $0.07924^{\{4\}}$ | $0.08412^{\{8\}}$ | $0.07817^{\{2\}}$ |
| | MRE | $\hat{a}$ | $0.27174^{\{3\}}$ | $0.28234^{\{9\}}$ | $0.27235^{\{4\}}$ | $0.27047^{\{2\}}$ | $0.27855^{\{5\}}$ | $0.2636^{\{1\}}$ | $0.28052^{\{6\}}$ | $0.28105^{\{7\}}$ | $0.28181^{\{8\}}$ |
| | | $\hat{b}$ | $0.31262^{\{5\}}$ | $0.31501^{\{7\}}$ | $0.33002^{\{9\}}$ | $0.31913^{\{8\}}$ | $0.3073^{\{2\}}$ | $0.31375^{\{6\}}$ | $0.31077^{\{3\}}$ | $0.30407^{\{1\}}$ | $0.31153^{\{4\}}$ |
| | | $\hat{\theta}$ | $0.24953^{\{3\}}$ | $0.26494^{\{9\}}$ | $0.26063^{\{8\}}$ | $0.25032^{\{5\}}$ | $0.25111^{\{6\}}$ | $0.24773^{\{1\}}$ | $0.24998^{\{4\}}$ | $0.25409^{\{7\}}$ | $0.24899^{\{2\}}$ |
| | $\sum Ranks$ | | $35.5^{\{2\}}$ | $74^{\{9\}}$ | $60^{\{8\}}$ | $41^{\{5.5\}}$ | $40^{\{3\}}$ | $27^{\{1\}}$ | $41^{\{5.5\}}$ | $46^{\{7\}}$ | $40.5^{\{4\}}$ |
| 250 | BIAS | $\hat{a}$ | $0.78611^{\{2\}}$ | $0.80316^{\{3\}}$ | $0.76803^{\{1\}}$ | $0.82119^{\{5\}}$ | $0.82709^{\{7\}}$ | $0.82313^{\{6\}}$ | $0.86141^{\{9\}}$ | $0.80404^{\{4\}}$ | $0.82822^{\{8\}}$ |
| | | $\hat{b}$ | $0.05488^{\{6\}}$ | $0.0553^{\{8\}}$ | $0.05237^{\{1\}}$ | $0.0549^{\{7\}}$ | $0.05434^{\{5\}}$ | $0.05403^{\{4\}}$ | $0.05328^{\{3\}}$ | $0.05624^{\{9\}}$ | $0.05282^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.20126^{\{7\}}$ | $0.19982^{\{4\}}$ | $0.18797^{\{1\}}$ | $0.20054^{\{6\}}$ | $0.19727^{\{2\}}$ | $0.20026^{\{5\}}$ | $0.19946^{\{3\}}$ | $0.20129^{\{8\}}$ | $0.20318^{\{9\}}$ |
| | MSE | $\hat{a}$ | $1.14053^{\{2\}}$ | $1.14561^{\{3\}}$ | $1.05426^{\{1\}}$ | $1.2305^{\{6\}}$ | $1.22893^{\{5\}}$ | $1.24908^{\{7\}}$ | $1.33801^{\{9\}}$ | $1.14669^{\{4\}}$ | $1.26124^{\{8\}}$ |
| | | $\hat{b}$ | $0.00471^{\{4\}}$ | $0.00492^{\{8\}}$ | $0.00454^{\{3\}}$ | $0.00479^{\{7\}}$ | $0.00475^{\{5\}}$ | $0.00476^{\{6\}}$ | $0.00444^{\{1\}}$ | $0.00504^{\{9\}}$ | $0.00449^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.06512^{\{8\}}$ | $0.06174^{\{2\}}$ | $0.05579^{\{1\}}$ | $0.06371^{\{7\}}$ | $0.06249^{\{3\}}$ | $0.06344^{\{5\}}$ | $0.06302^{\{4\}}$ | $0.06347^{\{6\}}$ | $0.0667^{\{9\}}$ |
| | MRE | $\hat{a}$ | $0.2246^{\{2\}}$ | $0.22947^{\{3\}}$ | $0.21944^{\{1\}}$ | $0.23462^{\{5\}}$ | $0.23631^{\{7\}}$ | $0.23518^{\{6\}}$ | $0.24612^{\{9\}}$ | $0.22973^{\{4\}}$ | $0.23664^{\{8\}}$ |
| | | $\hat{b}$ | $0.27438^{\{6\}}$ | $0.2765^{\{8\}}$ | $0.26183^{\{1\}}$ | $0.27449^{\{7\}}$ | $0.2717^{\{5\}}$ | $0.27017^{\{4\}}$ | $0.26642^{\{3\}}$ | $0.28122^{\{9\}}$ | $0.26412^{\{2\}}$ |
| | | $\hat{\theta}$ | $0.22362^{\{7\}}$ | $0.22203^{\{4\}}$ | $0.20885^{\{1\}}$ | $0.22282^{\{6\}}$ | $0.21919^{\{2\}}$ | $0.22251^{\{5\}}$ | $0.22162^{\{3\}}$ | $0.22365^{\{8\}}$ | $0.22575^{\{9\}}$ |
| | $\sum Ranks$ | | $44^{\{4.5\}}$ | $43^{\{3\}}$ | $11^{\{1\}}$ | $56^{\{7\}}$ | $41^{\{2\}}$ | $48^{\{6\}}$ | $44^{\{4.5\}}$ | $61^{\{9\}}$ | $57^{\{8\}}$ |
| 400 | BIAS | $\hat{a}$ | $0.72511^{\{8\}}$ | $0.69183^{\{5\}}$ | $0.72313^{\{7\}}$ | $0.68504^{\{2\}}$ | $0.72918^{\{9\}}$ | $0.71778^{\{6\}}$ | $0.66407^{\{1\}}$ | $0.68603^{\{3\}}$ | $0.68725^{\{4\}}$ |
| | | $\hat{b}$ | $0.05005^{\{8\}}$ | $0.05027^{\{9\}}$ | $0.0486^{\{4\}}$ | $0.04828^{\{3\}}$ | $0.04967^{\{6\}}$ | $0.0495^{\{5\}}$ | $0.05001^{\{7\}}$ | $0.04812^{\{2\}}$ | $0.0468^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.18303^{\{9\}}$ | $0.17924^{\{6\}}$ | $0.17796^{\{5\}}$ | $0.17493^{\{4\}}$ | $0.17927^{\{7\}}$ | $0.17934^{\{8\}}$ | $0.17413^{\{3\}}$ | $0.17355^{\{2\}}$ | $0.17003^{\{1\}}$ |
| | MSE | $\hat{a}$ | $0.93647^{\{6\}}$ | $0.89678^{\{5\}}$ | $0.9776^{\{8\}}$ | $0.86046^{\{2\}}$ | $0.98931^{\{9\}}$ | $0.97578^{\{7\}}$ | $0.78785^{\{1\}}$ | $0.8792^{\{4\}}$ | $0.87616^{\{3\}}$ |
| | | $\hat{b}$ | $0.00386^{\{5\}}$ | $0.00419^{\{9\}}$ | $0.00385^{\{4\}}$ | $0.00377^{\{2\}}$ | $0.004^{\{6\}}$ | $0.00403^{\{8\}}$ | $0.00401^{\{7\}}$ | $0.00384^{\{3\}}$ | $0.00356^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.05084^{\{8\}}$ | $0.05081^{\{7\}}$ | $0.04947^{\{5\}}$ | $0.04914^{\{4\}}$ | $0.05027^{\{6\}}$ | $0.05245^{\{9\}}$ | $0.0473^{\{2\}}$ | $0.04817^{\{3\}}$ | $0.04501^{\{1\}}$ |
| | MRE | $\hat{a}$ | $0.20717^{\{8\}}$ | $0.19766^{\{5\}}$ | $0.20661^{\{7\}}$ | $0.19573^{\{2\}}$ | $0.20834^{\{9\}}$ | $0.20508^{\{6\}}$ | $0.18973^{\{1\}}$ | $0.19601^{\{3\}}$ | $0.19636^{\{4\}}$ |
| | | $\hat{b}$ | $0.25025^{\{8\}}$ | $0.25137^{\{9\}}$ | $0.24302^{\{4\}}$ | $0.2414^{\{3\}}$ | $0.24833^{\{6\}}$ | $0.24748^{\{5\}}$ | $0.25003^{\{7\}}$ | $0.2406^{\{2\}}$ | $0.234^{\{1\}}$ |
| | | $\hat{\theta}$ | $0.20336^{\{9\}}$ | $0.19915^{\{6\}}$ | $0.19773^{\{5\}}$ | $0.19437^{\{4\}}$ | $0.19919^{\{7\}}$ | $0.19926^{\{8\}}$ | $0.19347^{\{3\}}$ | $0.19284^{\{2\}}$ | $0.18892^{\{1\}}$ |
| | $\sum Ranks$ | | $69^{\{9\}}$ | $61^{\{6\}}$ | $49^{\{5\}}$ | $26^{\{3\}}$ | $65^{\{8\}}$ | $62^{\{7\}}$ | $32^{\{4\}}$ | $24^{\{2\}}$ | $17^{\{1\}}$ |
| 600 | BIAS | $\hat{a}$ | $0.59176^{\{7\}}$ | $0.58586^{\{5\}}$ | $0.58491^{\{4\}}$ | $0.60071^{\{8\}}$ | $0.55951^{\{1\}}$ | $0.58991^{\{6\}}$ | $0.60198^{\{9\}}$ | $0.57781^{\{2\}}$ | $0.58391^{\{3\}}$ |
| | | $\hat{b}$ | $0.04164^{\{8\}}$ | $0.04162^{\{7\}}$ | $0.04131^{\{6\}}$ | $0.04292^{\{9\}}$ | $0.0403^{\{3\}}$ | $0.04096^{\{4\}}$ | $0.04025^{\{2\}}$ | $0.03995^{\{1\}}$ | $0.04126^{\{5\}}$ |
| | | $\hat{\theta}$ | $0.15248^{\{8\}}$ | $0.15082^{\{6\}}$ | $0.14843^{\{4\}}$ | $0.15406^{\{9\}}$ | $0.14766^{\{2\}}$ | $0.15121^{\{7\}}$ | $0.14786^{\{3\}}$ | $0.14765^{\{1\}}$ | $0.14985^{\{5\}}$ |
| | MSE | $\hat{a}$ | $0.65994^{\{6\}}$ | $0.63873^{\{4\}}$ | $0.64113^{\{5\}}$ | $0.67208^{\{8\}}$ | $0.58479^{\{1\}}$ | $0.69661^{\{9\}}$ | $0.67053^{\{7\}}$ | $0.61673^{\{2\}}$ | $0.62322^{\{3\}}$ |
| | | $\hat{b}$ | $0.00288^{\{8\}}$ | $0.00286^{\{6.5\}}$ | $0.00282^{\{5\}}$ | $0.00306^{\{9\}}$ | $0.00266^{\{2\}}$ | $0.00272^{\{3.5\}}$ | $0.00272^{\{3.5\}}$ | $0.00263^{\{1\}}$ | $0.00286^{\{6.5\}}$ |
| | | $\hat{\theta}$ | $0.03752^{\{8\}}$ | $0.03636^{\{4\}}$ | $0.0354^{\{3\}}$ | $0.03817^{\{9\}}$ | $0.03495^{\{2\}}$ | $0.03706^{\{7\}}$ | $0.03646^{\{5.5\}}$ | $0.03476^{\{1\}}$ | $0.03646^{\{5.5\}}$ |
| | MRE | $\hat{a}$ | $0.16907^{\{7\}}$ | $0.16739^{\{5\}}$ | $0.16712^{\{4\}}$ | $0.17163^{\{8\}}$ | $0.15986^{\{1\}}$ | $0.16855^{\{6\}}$ | $0.17199^{\{9\}}$ | $0.16509^{\{2\}}$ | $0.16683^{\{3\}}$ |
| | | $\hat{b}$ | $0.20821^{\{8\}}$ | $0.20811^{\{7\}}$ | $0.20656^{\{6\}}$ | $0.21459^{\{9\}}$ | $0.20148^{\{3\}}$ | $0.20479^{\{4\}}$ | $0.20125^{\{2\}}$ | $0.19973^{\{1\}}$ | $0.20631^{\{5\}}$ |
| | | $\hat{\theta}$ | $0.16942^{\{8\}}$ | $0.16758^{\{6\}}$ | $0.16492^{\{4\}}$ | $0.17118^{\{9\}}$ | $0.16407^{\{2\}}$ | $0.16801^{\{7\}}$ | $0.16429^{\{3\}}$ | $0.16405^{\{1\}}$ | $0.1665^{\{5\}}$ |
| | $\sum Ranks$ | | $68^{\{8\}}$ | $50.5^{\{6\}}$ | $41^{\{3.5\}}$ | $78^{\{9\}}$ | $17^{\{2\}}$ | $53.5^{\{7\}}$ | $44^{\{5\}}$ | $12^{\{1\}}$ | $41^{\{3.5\}}$ |

Table 3.5: BIAS, MSE and MRE for (a=3.0, b=0.9, theta=0.2)

| n | Est. | Est. Par. | MLE | ADE | CVME | MPSE | LSE | RTADE | WLSE | MSADE | MSALDE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 35 | BIAS | $\hat{a}$ | 0.84106$^{\{4\}}$ | 0.86579$^{\{8\}}$ | 0.83381$^{\{3\}}$ | 0.82683$^{\{2\}}$ | 0.82575$^{\{1\}}$ | 0.84176$^{\{5\}}$ | 0.89041$^{\{9\}}$ | 0.86449$^{\{7\}}$ | 0.85383$^{\{6\}}$ |
| | | $\hat{b}$ | 0.32947$^{\{4\}}$ | 0.33016$^{\{5\}}$ | 0.32317$^{\{2\}}$ | 0.32657$^{\{3\}}$ | 0.33596$^{\{6\}}$ | 0.34315$^{\{9\}}$ | 0.3215$^{\{1\}}$ | 0.34172$^{\{8\}}$ | 0.33788$^{\{7\}}$ |
| | | $\hat{\theta}$ | 0.7355$^{\{3\}}$ | 0.75642$^{\{7\}}$ | 0.75302$^{\{5\}}$ | 0.73018$^{\{2\}}$ | 0.73659$^{\{4\}}$ | 0.79221$^{\{9\}}$ | 0.72242$^{\{1\}}$ | 0.78203$^{\{8\}}$ | 0.75503$^{\{6\}}$ |
| | MSE | $\hat{a}$ | 1.18909$^{\{3\}}$ | 1.25316$^{\{8\}}$ | 1.1971$^{\{4\}}$ | 1.16063$^{\{1\}}$ | 1.16591$^{\{2\}}$ | 1.22341$^{\{6\}}$ | 1.29588$^{\{9\}}$ | 1.21723$^{\{5\}}$ | 1.2239$^{\{7\}}$ |
| | | $\hat{b}$ | 0.14457$^{\{3\}}$ | 0.14553$^{\{5\}}$ | 0.13804$^{\{1\}}$ | 0.14467$^{\{4\}}$ | 0.14844$^{\{6\}}$ | 0.15517$^{\{9\}}$ | 0.1395$^{\{2\}}$ | 0.15272$^{\{8\}}$ | 0.15207$^{\{7\}}$ |
| | | $\hat{\theta}$ | 0.83556$^{\{3\}}$ | 0.8659$^{\{6\}}$ | 0.86107$^{\{5\}}$ | 0.84557$^{\{4\}}$ | 0.83488$^{\{2\}}$ | 0.92984$^{\{9\}}$ | 0.80528$^{\{1\}}$ | 0.91477$^{\{8\}}$ | 0.86743$^{\{7\}}$ |
| | MRE | $\hat{a}$ | 0.28035$^{\{4\}}$ | 0.2886$^{\{8\}}$ | 0.27794$^{\{3\}}$ | 0.27561$^{\{2\}}$ | 0.27525$^{\{1\}}$ | 0.28059$^{\{5\}}$ | 0.2968$^{\{9\}}$ | 0.28816$^{\{7\}}$ | 0.28461$^{\{6\}}$ |
| | | $\hat{b}$ | 0.36608$^{\{4\}}$ | 0.36684$^{\{5\}}$ | 0.35908$^{\{2\}}$ | 0.36285$^{\{3\}}$ | 0.37329$^{\{6\}}$ | 0.38128$^{\{9\}}$ | 0.35722$^{\{1\}}$ | 0.37969$^{\{8\}}$ | 0.37542$^{\{7\}}$ |
| | | $\hat{\theta}$ | 0.36775$^{\{3\}}$ | 0.37821$^{\{7\}}$ | 0.37651$^{\{5\}}$ | 0.36509$^{\{2\}}$ | 0.36829$^{\{4\}}$ | 0.3961$^{\{9\}}$ | 0.36121$^{\{1\}}$ | 0.39101$^{\{8\}}$ | 0.37751$^{\{6\}}$ |
| | $\sum Ranks$ | | 31$^{\{3\}}$ | 59$^{\{6.5\}}$ | 30$^{\{2\}}$ | 23$^{\{1\}}$ | 32$^{\{4\}}$ | 70$^{\{9\}}$ | 34$^{\{5\}}$ | 67$^{\{8\}}$ | 59$^{\{6.5\}}$ |
| 100 | BIAS | $\hat{a}$ | 0.59279$^{\{5\}}$ | 0.61825$^{\{8\}}$ | 0.59097$^{\{2\}}$ | 0.60289$^{\{7\}}$ | 0.58279$^{\{1\}}$ | 0.62448$^{\{9\}}$ | 0.59122$^{\{3\}}$ | 0.60016$^{\{6\}}$ | 0.59234$^{\{4\}}$ |
| | | $\hat{b}$ | 0.3133$^{\{2\}}$ | 0.31989$^{\{6\}}$ | 0.30745$^{\{1\}}$ | 0.32136$^{\{7\}}$ | 0.32607$^{\{9\}}$ | 0.3246$^{\{8\}}$ | 0.316$^{\{3\}}$ | 0.31751$^{\{5\}}$ | 0.31681$^{\{4\}}$ |
| | | $\hat{\theta}$ | 0.67705$^{\{2\}}$ | 0.71732$^{\{9\}}$ | 0.67169$^{\{1\}}$ | 0.69994$^{\{5\}}$ | 0.70508$^{\{6\}}$ | 0.71555$^{\{8\}}$ | 0.71066$^{\{7\}}$ | 0.68141$^{\{3\}}$ | 0.68434$^{\{4\}}$ |
| | MSE | $\hat{a}$ | 0.65735$^{\{4\}}$ | 0.72247$^{\{9\}}$ | 0.66182$^{\{6\}}$ | 0.69203$^{\{7\}}$ | 0.62598$^{\{1\}}$ | 0.71433$^{\{8\}}$ | 0.64943$^{\{3\}}$ | 0.64631$^{\{2\}}$ | 0.65928$^{\{5\}}$ |
| | | $\hat{b}$ | 0.13265$^{\{2\}}$ | 0.14146$^{\{7\}}$ | 0.13124$^{\{1\}}$ | 0.14095$^{\{6\}}$ | 0.14536$^{\{9\}}$ | 0.14172$^{\{8\}}$ | 0.13432$^{\{3\}}$ | 0.13654$^{\{5\}}$ | 0.13457$^{\{4\}}$ |
| | | $\hat{\theta}$ | 0.71118$^{\{1\}}$ | 0.80857$^{\{9\}}$ | 0.71727$^{\{2\}}$ | 0.75761$^{\{5\}}$ | 0.77219$^{\{6\}}$ | 0.80166$^{\{8\}}$ | 0.78422$^{\{7\}}$ | 0.7304$^{\{3\}}$ | 0.73835$^{\{4\}}$ |
| | MRE | $\hat{a}$ | 0.1976$^{\{5\}}$ | 0.20608$^{\{8\}}$ | 0.19699$^{\{2\}}$ | 0.20096$^{\{7\}}$ | 0.19426$^{\{1\}}$ | 0.20816$^{\{9\}}$ | 0.19707$^{\{3\}}$ | 0.20005$^{\{6\}}$ | 0.19745$^{\{4\}}$ |
| | | $\hat{b}$ | 0.34811$^{\{2\}}$ | 0.35543$^{\{6\}}$ | 0.34161$^{\{1\}}$ | 0.35707$^{\{7\}}$ | 0.3623$^{\{9\}}$ | 0.36067$^{\{8\}}$ | 0.35111$^{\{3\}}$ | 0.35279$^{\{5\}}$ | 0.35201$^{\{4\}}$ |
| | | $\hat{\theta}$ | 0.33853$^{\{2\}}$ | 0.35866$^{\{9\}}$ | 0.33585$^{\{1\}}$ | 0.34997$^{\{5\}}$ | 0.35254$^{\{6\}}$ | 0.35777$^{\{8\}}$ | 0.35533$^{\{7\}}$ | 0.3407$^{\{3\}}$ | 0.34217$^{\{4\}}$ |
| | $\sum Ranks$ | | 25$^{\{2\}}$ | 71$^{\{8\}}$ | 17$^{\{1\}}$ | 56$^{\{7\}}$ | 48$^{\{6\}}$ | 74$^{\{9\}}$ | 39$^{\{5\}}$ | 38$^{\{4\}}$ | 37$^{\{3\}}$ |
| 250 | BIAS | $\hat{a}$ | 0.39662$^{\{2\}}$ | 0.4051$^{\{6\}}$ | 0.39933$^{\{3\}}$ | 0.42327$^{\{8\}}$ | 0.40451$^{\{5\}}$ | 0.39937$^{\{4\}}$ | 0.41941$^{\{7\}}$ | 0.38515$^{\{1\}}$ | 0.42853$^{\{9\}}$ |
| | | $\hat{b}$ | 0.29421$^{\{8\}}$ | 0.29096$^{\{5\}}$ | 0.29122$^{\{6\}}$ | 0.29928$^{\{9\}}$ | 0.28822$^{\{4\}}$ | 0.28672$^{\{3\}}$ | 0.28541$^{\{2\}}$ | 0.27759$^{\{1\}}$ | 0.29164$^{\{7\}}$ |
| | | $\hat{\theta}$ | 0.57047$^{\{3\}}$ | 0.5736$^{\{4\}}$ | 0.57915$^{\{6\}}$ | 0.59318$^{\{8\}}$ | 0.57607$^{\{5\}}$ | 0.56211$^{\{2\}}$ | 0.58368$^{\{7\}}$ | 0.55989$^{\{1\}}$ | 0.59556$^{\{9\}}$ |
| | MSE | $\hat{a}$ | 0.30221$^{\{2\}}$ | 0.30551$^{\{4\}}$ | 0.30311$^{\{3\}}$ | 0.35154$^{\{9\}}$ | 0.31591$^{\{6\}}$ | 0.30607$^{\{5\}}$ | 0.33139$^{\{7\}}$ | 0.27854$^{\{1\}}$ | 0.34993$^{\{8\}}$ |
| | | $\hat{b}$ | 0.12537$^{\{8\}}$ | 0.12205$^{\{7\}}$ | 0.12146$^{\{6\}}$ | 0.1263$^{\{9\}}$ | 0.11979$^{\{5\}}$ | 0.11895$^{\{4\}}$ | 0.11488$^{\{2\}}$ | 0.11025$^{\{1\}}$ | 0.11808$^{\{3\}}$ |
| | | $\hat{\theta}$ | 0.53497$^{\{3\}}$ | 0.53952$^{\{4\}}$ | 0.54302$^{\{5\}}$ | 0.56407$^{\{9\}}$ | 0.55325$^{\{7\}}$ | 0.51686$^{\{1\}}$ | 0.55198$^{\{6\}}$ | 0.51757$^{\{2\}}$ | 0.56229$^{\{8\}}$ |
| | MRE | $\hat{a}$ | 0.13221$^{\{2\}}$ | 0.13503$^{\{6\}}$ | 0.13311$^{\{3\}}$ | 0.14109$^{\{8\}}$ | 0.13484$^{\{5\}}$ | 0.13312$^{\{4\}}$ | 0.1398$^{\{7\}}$ | 0.12838$^{\{1\}}$ | 0.14284$^{\{9\}}$ |
| | | $\hat{b}$ | 0.3269$^{\{8\}}$ | 0.32329$^{\{5\}}$ | 0.32357$^{\{6\}}$ | 0.33253$^{\{9\}}$ | 0.32025$^{\{4\}}$ | 0.31858$^{\{3\}}$ | 0.31712$^{\{2\}}$ | 0.30844$^{\{1\}}$ | 0.32405$^{\{7\}}$ |
| | | $\hat{\theta}$ | 0.28524$^{\{3\}}$ | 0.2868$^{\{4\}}$ | 0.28958$^{\{6\}}$ | 0.29659$^{\{8\}}$ | 0.28803$^{\{5\}}$ | 0.28105$^{\{2\}}$ | 0.29184$^{\{7\}}$ | 0.27995$^{\{1\}}$ | 0.29778$^{\{9\}}$ |
| | $\sum Ranks$ | | 39$^{\{3\}}$ | 45$^{\{5\}}$ | 44$^{\{4\}}$ | 77$^{\{9\}}$ | 46$^{\{6\}}$ | 28$^{\{2\}}$ | 47$^{\{7\}}$ | 10$^{\{1\}}$ | 69$^{\{8\}}$ |
| 400 | BIAS | $\hat{a}$ | 0.30222$^{\{2\}}$ | 0.33022$^{\{9\}}$ | 0.31215$^{\{6\}}$ | 0.30516$^{\{3\}}$ | 0.31041$^{\{5\}}$ | 0.30617$^{\{4\}}$ | 0.31526$^{\{7\}}$ | 0.29711$^{\{1\}}$ | 0.32329$^{\{8\}}$ |
| | | $\hat{b}$ | 0.27488$^{\{7\}}$ | 0.26769$^{\{4\}}$ | 0.26906$^{\{5\}}$ | 0.26205$^{\{1\}}$ | 0.27651$^{\{9\}}$ | 0.27337$^{\{6\}}$ | 0.26712$^{\{3\}}$ | 0.2639$^{\{2\}}$ | 0.27532$^{\{8\}}$ |
| | | $\hat{\theta}$ | 0.49448$^{\{4\}}$ | 0.50706$^{\{8\}}$ | 0.4965$^{\{5\}}$ | 0.48847$^{\{3\}}$ | 0.49784$^{\{6\}}$ | 0.48449$^{\{2\}}$ | 0.5015$^{\{7\}}$ | 0.46853$^{\{1\}}$ | 0.51461$^{\{9\}}$ |
| | MSE | $\hat{a}$ | 0.17002$^{\{2\}}$ | 0.20461$^{\{9\}}$ | 0.18494$^{\{6\}}$ | 0.1718$^{\{3\}}$ | 0.18208$^{\{5\}}$ | 0.17697$^{\{4\}}$ | 0.19295$^{\{8\}}$ | 0.16879$^{\{1\}}$ | 0.19085$^{\{7\}}$ |
| | | $\hat{b}$ | 0.1126$^{\{8\}}$ | 0.1031$^{\{2\}}$ | 0.10766$^{\{5\}}$ | 0.10205$^{\{1\}}$ | 0.11365$^{\{9\}}$ | 0.11236$^{\{7\}}$ | 0.10415$^{\{3\}}$ | 0.10554$^{\{4\}}$ | 0.1092$^{\{6\}}$ |
| | | $\hat{\theta}$ | 0.39733$^{\{4\}}$ | 0.40919$^{\{7\}}$ | 0.40055$^{\{6\}}$ | 0.38915$^{\{3\}}$ | 0.39882$^{\{5\}}$ | 0.38375$^{\{2\}}$ | 0.41294$^{\{8\}}$ | 0.36142$^{\{1\}}$ | 0.42418$^{\{9\}}$ |
| | MRE | $\hat{a}$ | 0.10074$^{\{2\}}$ | 0.11007$^{\{9\}}$ | 0.10405$^{\{6\}}$ | 0.10172$^{\{3\}}$ | 0.10347$^{\{5\}}$ | 0.10206$^{\{4\}}$ | 0.10509$^{\{7\}}$ | 0.09904$^{\{1\}}$ | 0.10776$^{\{8\}}$ |
| | | $\hat{b}$ | 0.30542$^{\{7\}}$ | 0.29743$^{\{4\}}$ | 0.29895$^{\{5\}}$ | 0.29116$^{\{1\}}$ | 0.30724$^{\{9\}}$ | 0.30374$^{\{6\}}$ | 0.29679$^{\{3\}}$ | 0.29322$^{\{2\}}$ | 0.30592$^{\{8\}}$ |
| | | $\hat{\theta}$ | 0.24724$^{\{4\}}$ | 0.25353$^{\{8\}}$ | 0.24825$^{\{5\}}$ | 0.24423$^{\{3\}}$ | 0.24892$^{\{6\}}$ | 0.24224$^{\{2\}}$ | 0.25075$^{\{7\}}$ | 0.23426$^{\{1\}}$ | 0.2573$^{\{9\}}$ |
| | $\sum Ranks$ | | 40$^{\{4\}}$ | 60$^{\{8\}}$ | 49$^{\{5\}}$ | 21$^{\{2\}}$ | 59$^{\{7\}}$ | 37$^{\{3\}}$ | 53$^{\{6\}}$ | 14$^{\{1\}}$ | 72$^{\{9\}}$ |
| 600 | BIAS | $\hat{a}$ | 0.24666$^{\{2\}}$ | 0.25303$^{\{9\}}$ | 0.24706$^{\{3\}}$ | 0.24525$^{\{1\}}$ | 0.25105$^{\{8\}}$ | 0.25007$^{\{5\}}$ | 0.25015$^{\{6\}}$ | 0.2471$^{\{4\}}$ | 0.25027$^{\{7\}}$ |
| | | $\hat{b}$ | 0.25303$^{\{8\}}$ | 0.2493$^{\{6\}}$ | 0.25202$^{\{7\}}$ | 0.24822$^{\{3\}}$ | 0.2477$^{\{2\}}$ | 0.24892$^{\{5\}}$ | 0.24884$^{\{4\}}$ | 0.2476$^{\{1\}}$ | 0.25639$^{\{9\}}$ |
| | | $\hat{\theta}$ | 0.42389$^{\{6\}}$ | 0.42183$^{\{4\}}$ | 0.42726$^{\{8\}}$ | 0.41651$^{\{1\}}$ | 0.4262$^{\{7\}}$ | 0.41882$^{\{2\}}$ | 0.42371$^{\{5\}}$ | 0.42089$^{\{3\}}$ | 0.43371$^{\{9\}}$ |
| | MSE | $\hat{a}$ | 0.11338$^{\{7\}}$ | 0.12115$^{\{9\}}$ | 0.10938$^{\{1\}}$ | 0.10992$^{\{2\}}$ | 0.11256$^{\{4\}}$ | 0.11593$^{\{8\}}$ | 0.11162$^{\{3\}}$ | 0.11312$^{\{6\}}$ | 0.1129$^{\{5\}}$ |
| | | $\hat{b}$ | 0.1021$^{\{8\}}$ | 0.09816$^{\{7\}}$ | 0.09746$^{\{6\}}$ | 0.09544$^{\{4\}}$ | 0.09329$^{\{1\}}$ | 0.09714$^{\{5\}}$ | 0.0946$^{\{3\}}$ | 0.09418$^{\{2\}}$ | 0.10292$^{\{9\}}$ |
| | | $\hat{\theta}$ | 0.29258$^{\{8\}}$ | 0.28816$^{\{5\}}$ | 0.28676$^{\{4\}}$ | 0.27799$^{\{1\}}$ | 0.2905$^{\{7\}}$ | 0.28128$^{\{2\}}$ | 0.2892$^{\{6\}}$ | 0.28308$^{\{3\}}$ | 0.29795$^{\{9\}}$ |
| | MRE | $\hat{a}$ | 0.08222$^{\{2\}}$ | 0.08434$^{\{9\}}$ | 0.08235$^{\{3\}}$ | 0.08175$^{\{1\}}$ | 0.08368$^{\{8\}}$ | 0.08336$^{\{5\}}$ | 0.08338$^{\{6\}}$ | 0.08237$^{\{4\}}$ | 0.08342$^{\{7\}}$ |
| | | $\hat{b}$ | 0.28114$^{\{8\}}$ | 0.277$^{\{6\}}$ | 0.28002$^{\{7\}}$ | 0.2758$^{\{3\}}$ | 0.27522$^{\{2\}}$ | 0.27658$^{\{5\}}$ | 0.27649$^{\{4\}}$ | 0.27511$^{\{1\}}$ | 0.28488$^{\{9\}}$ |
| | | $\hat{\theta}$ | 0.21194$^{\{6\}}$ | 0.21091$^{\{4\}}$ | 0.21363$^{\{8\}}$ | 0.20826$^{\{1\}}$ | 0.2131$^{\{7\}}$ | 0.20941$^{\{2\}}$ | 0.21185$^{\{5\}}$ | 0.21044$^{\{3\}}$ | 0.21685$^{\{9\}}$ |
| | $\sum Ranks$ | | 55$^{\{7\}}$ | 59$^{\{8\}}$ | 47$^{\{6\}}$ | 17$^{\{1\}}$ | 46$^{\{5\}}$ | 39$^{\{3\}}$ | 42$^{\{4\}}$ | 27$^{\{2\}}$ | 73$^{\{9\}}$ |

Table 3.6: Partial and overall ranks of all estimation methods.

| Parameter | $n$ | MLE | ADE | CVME | MPSE | LSE | RTADE | WLSE | MSADE | MSALDE |
|---|---|---|---|---|---|---|---|---|---|---|
| $a = 1.5, b = 0.5,\ \theta = 0.25$ | 35 | 8.0 | 5.0 | 4.0 | 6.0 | 7.0 | 2.0 | 1.0 | 9.0 | 3.0 |
| | 100 | 7.0 | 6.0 | 1.0 | 8.0 | 2.0 | 3.0 | 5.0 | 9.0 | 4.0 |
| | 250 | 6.0 | 7.0 | 4.0 | 5.0 | 9.0 | 3.0 | 1.0 | 8.0 | 2.0 |
| | 400 | 1.0 | 7.0 | 4.0 | 2.5 | 9.0 | 6.0 | 2.5 | 5.0 | 8.0 |
| | 600 | 1.0 | 6.5 | 4.5 | 2.0 | 8.0 | 3.0 | 4.5 | 9.0 | 6.5 |
| $a = 2.5, b = 0.75,\ \theta = 1.5$ | 35 | 4.0 | 7.0 | 6.0 | 5.0 | 3.0 | 9.0 | 8.0 | 2.0 | 1.0 |
| | 100 | 3.0 | 4.0 | 5.5 | 5.5 | 2.0 | 9.0 | 8.0 | 7.0 | 1.0 |
| | 250 | 7.0 | 3.0 | 8.0 | 4.0 | 1.5 | 9.0 | 1.5 | 5.0 | 6.0 |
| | 400 | 3.0 | 8.0 | 4.0 | 7.0 | 2.0 | 9.0 | 1.0 | 6.0 | 5.0 |
| | 600 | 5.0 | 6.0 | 3.0 | 8.0 | 9.0 | 2.0 | 1.0 | 4.0 | 7.0 |
| $a = 2.0, b = 1.5,\ \theta = 0.5$ | 35 | 6.0 | 4.0 | 3.0 | 9.0 | 5.0 | 8.0 | 7.0 | 2.0 | 1.0 |
| | 100 | 8.0 | 4.0 | 1.5 | 7.0 | 5.5 | 5.5 | 9.0 | 3.0 | 1.5 |
| | 250 | 6.0 | 4.0 | 3.0 | 5.0 | 7.0 | 9.0 | 8.0 | 1.0 | 2.0 |
| | 400 | 1.5 | 1.5 | 6.0 | 8.0 | 9.0 | 5.0 | 4.0 | 3.0 | 7.0 |
| | 600 | 7.0 | 3.0 | 6.0 | 4.0 | 5.0 | 2.0 | 8.5 | 8.5 | 1.0 |
| $a = 3.5, b = 0.2,\ \theta = 0.9$ | 35 | 7.0 | 4.0 | 3.0 | 8.0 | 1.0 | 9.0 | 5.5 | 5.5 | 2.0 |
| | 100 | 2.0 | 9.0 | 8.0 | 5.5 | 3.0 | 1.0 | 5.5 | 7.0 | 4.0 |
| | 250 | 4.5 | 3.0 | 1.0 | 7.0 | 2.0 | 6.0 | 4.5 | 9.0 | 8.0 |
| | 400 | 9.0 | 6.0 | 5.0 | 3.0 | 8.0 | 7.0 | 4.0 | 2.0 | 1.0 |
| | 600 | 8.0 | 6.0 | 3.5 | 9.0 | 2.0 | 7.0 | 5.0 | 1.0 | 3.5 |
| $a = 3.0, b = 0.9,\ \theta = 0.2$ | 35 | 3.0 | 6.5 | 2.0 | 1.0 | 4.0 | 9.0 | 5.0 | 8.0 | 6.5 |
| | 100 | 2.0 | 8.0 | 1.0 | 7.0 | 6.0 | 9.0 | 5.0 | 4.0 | 3.0 |
| | 250 | 3.0 | 5.0 | 4.0 | 9.0 | 6.0 | 2.0 | 7.0 | 1.0 | 8.0 |
| | 400 | 4.0 | 8.0 | 5.0 | 2.0 | 7.0 | 3.0 | 6.0 | 1.0 | 9.0 |
| | 600 | 7.0 | 8.0 | 6.0 | 1.0 | 5.0 | 3.0 | 4.0 | 2.0 | 9.0 |
| $\sum$ Ranks | | 123.0 | 139.5 | 102.0 | 138.5 | 128.0 | 140.5 | 121.5 | 122.0 | 110.0 |
| Overall Rank | | 5 | 8 | 1 | 7 | 6 | 9 | 3 | 4 | 2 |

Table 3.7: The ML estimates,-2 log-L, AIC, BIC, and CAIC for data set 1.

| $Model$ | $a$ | $b$ | $\theta$ | $\gamma$ | $AIC$ | $BIC$ | $-2L$ | $AICC$ |
|---|---|---|---|---|---|---|---|---|
| exponential | / | / | 0.4166675 | / | 182.045 | 183.9162 | 180.045 | 182.132 |
| Lindley | / | / | 0.6666664 | / | 166.2406 | 168.1118 | 164.2406 | 166.3275 |
| XLindley | / | / | 0.5772491 | / | 174.4943 | 176.3655 | 172.4943 | 174.5812 |
| new XLindley | / | / | 0.670488 | / | 170.271 | 172.1422 | 168.271 | 170.3579 |
| Xgamma | / | / | 0.9279678 | / | 169.8614 | 171.7326 | 167.8614 | 169.9484 |
| Zeghdoudi | / | / | 1.101979 | / | 138.8313 | 140.7025 | 136.8313 | 138.9182 |
| two-parameter Lindley I | / | / | 0.8546036 | 8.38814e-05 | 149.545 | 153.2874 | 145.545 | 149.8117 |
| gamma Lindley | / | / | 0.8314911 | 42.62868 | 149.9641 | 153.7065 | 145.9641 | 150.2308 |
| new quasi Lindley | / | / | 0.8248596 | 34.76987 | 150.3554 | 154.0978 | 146.3554 | 150.6221 |
| two-parameter Lindley II | / | / | 0.8281973 | 42.72697 | 150.345 | 154.0874 | 146.345 | 150.6116 |
| Power XLindley | / | / | 1.599186 | 0.4308425 | 291.5391 | 295.2815 | 287.5391 | 291.8058 |
| beta-Lindley | 18.89298 | 4.256092 | 1.08148 | / | 82.8295 | 88.44311 | 76.8295 | 83.37496 |
| beta-exponential | 25.13779 | 5.168618 | 0.7716722 | / | 82.82384 | 88.43744 | 76.82384 | 83.3693 |
| Chen | / | / | 0.02940777 | 1.288413 | 91.30852 | 95.05093 | 87.30852 | 91.57519 |
| beta-NewXLindley | 16.68304 | 7.036943 | 0.8074539 | / | 82.77138 | 88.38499 | 76.77138 | 83.31684 |

Table 3.8: The ML estimates,-2 log-L, AIC, BIC, and CAIC for data set 2.

| $Model$ | $a$ | $b$ | $\theta$ | $\gamma$ | $AIC$ | $BIC$ | $-2L$ | $AICC$ |
|---|---|---|---|---|---|---|---|---|
| exponential | / | / | 0.1989245 | / | 388.9947 | 391.2987 | 386.9947 | 389.0502 |
| Lindley | / | / | 0.3466344 | / | 367.9865 | 370.2906 | 365.9865 | 368.0421 |
| XLindley | / | / | 0.3128214 | / | 374.8971 | 377.2012 | 372.8971 | 374.9527 |
| new XLindley | / | / | 0.311163 | / | 377.825 | 380.129 | 375.825 | 377.8805 |
| Xgamma | / | / | 0.4833675 | / | 370.3286 | 372.6327 | 368.3286 | 370.3841 |
| Zeghdoudi | / | / | 0.5536467 | / | 351.3284 | 353.6325 | 349.3284 | 351.384 |
| two-parameter Lindley I | / | / | 0.3961074 | 0.000256688 | 358.5446 | 363.1527 | 354.5446 | 358.7136 |
| gamma Lindley | / | / | 0.3965907 | 58.91614 | 358.7517 | 363.3598 | 354.7517 | 358.9207 |
| new quasi Lindley | / | / | 0.3966442 | 46.35307 | 358.688 | 363.2962 | 354.688 | 358.8571 |
| two-parameter Lindley II | / | / | 0.3963045 | 67.04979 | 358.7988 | 363.4069 | 354.7988 | 358.9678 |
| Power XLindley | / | / | 1.413932 | 0.3184585 | 576.7272 | 581.3353 | 572.7272 | 576.8962 |
| beta-Lindley | 55.89799 | 0.1041794 | 3.051239 | / | 342.486 | 349.3982 | 336.486 | 342.8289 |
| beta-exponential | 121.4675 | 0.101828 | 3.009173 | / | 341.4334 | 348.3456 | 335.4334 | 341.7763 |
| Chen | / | / | 0.0439729 | 0.6392914 | 357.2962 | 361.9043 | 353.2962 | 357.4652 |
| beta-NewXLindley | 99.46829 | 0.08597327 | 3.779259 | / | 340.5521 | 347.4643 | 334.5521 | 340.895 |

# Conclusion

In order to address the shortcomings of current models in handling various types of survival and reliability data, this thesis has introduced and investigated three new probability distributions: the truncated new-XLindley distribution, the two-parameter beta-exponential distribution, and the beta new-XLindley distribution. These distributions provided improved flexibility and interpretability and were built within the beta-generated family framework.

For every model, a comprehensive theoretical analysis covering fundamental statistical properties like moments, quantile functions, order statistics, and reliability measures was carried out. Monte Carlo simulations were used to assess the effectiveness of various estimation techniques, most notably maximum likelihood estimation and Bayesian estimation. The efficacy of the models was further illustrated by their applications to real-world data sets, where they outperformed a number of traditional and modern models according to metrics like AIC and BIC.

Given the frequency of truncated data in real-world contexts, the inclusion of the truncated new-XLindley distribution is especially important. With its introduction, the new-XLindley family's practical relevance is extended to scenarios in which truncation mechanisms result in the availability of only partial data.

**Perspectives**

This work opens up several research directions, including:

- Multivariate Extensions: By extending the suggested distributions to the multivariate case, more thorough models for reliability systems with numerous components or joint survival times may be produced.

- Regression Models: By incorporating covariates using survival regression frameworks or generalized linear models, their applicability in disciplines like economics, engineering, and medicine can be expanded.

- Bayesian Hierarchical Models: More research in this area, especially when employing Markov Chain Monte Carlo (MCMC) techniques and hierarchical modeling, may provide a better understanding of parameter behavior and predictive distributions.
- Data with Censored and Competing Risks: The models can be modified to manage scenarios with competing risks or censored data, which are prevalent in studies on industrial and medical reliability.
- Software Implementation: Practitioners and researchers would be more likely to adopt these new distributions if R or Python packages were developed for them.
- Diagnostic and Goodness-of-Fit Tools: More reliable statistical tests and diagnostic plots tailored to these distributions would help in the selection and validation of useful models.

In conclusion, the suggested models improve upon existing statistical approaches for survival and reliability data and open up a wide range of new research avenues with the goal of extending their theoretical reach and applicability in real-world settings.

## 3.5 Annexe

### 3.5.1 Model selection with different criteria

When we have models estimated by a maximum likelihood method, the likelihood ratio test is often used to compare these models two by two. It is only applicable to nested models (derived from each other by adding or removing terms), and it is assumed that the two models compared fit the data correctly. When many models have to be compared with each other, the risk of rejecting the null hypothesis when it is true increases. To solve this problem, there are solutions to compare the models, like the AIC (Akaike information criterion), BIC (Bayesian Information Criterion), -2L (-2Log-Likelihood), and AICC (Consistent Akaike Information Criterion).

### Akaike information criterion

Le critère d'information d'Akaike est donnée par:

$$AIC = -2\log(L) + 2k,$$

where $L$ is the maximized likelihood and $k$ is the number of parameters in the model. With this criterion, the deviance of the model ($-2log(L)$) is penalized by 2 times the number of parameters. The AIC therefore represents a compromise between bias (which decreases with the number of parameters) and parsimony (need to describe the data with the smallest possible number of parameters).

-Rigor would require that all the compared models all derive from the same "complete" included in the list of compared models.

-It is necessary to verify that this "complete" model correctly fits the data.

-The best model is the one with the lowest AIC.

### Consistent Akaike information criterion

When the number of parameters $k$ is large compared to the number of observations $n$, that is to say if $n/k < 40$, it is recommended to use the corrected AIC.

The corrected Akaike information criterion AICc, is defined by:

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}.$$

### Bayesian Information Criterion

In the case where we have several models that are very close to each other, and it is difficult to decide which of them is really the best. One possible approach is to use all of these models to make the inferences.

For this purpose, the current trend is rather to rely on the BIC.

The Bayesian information criterion BIC is defined by:

$$BIC = -2\log(L) + k\log(n).$$

The BIC was discussed [32] to select models in the case of large samples (several thousand observations) for which the AIC and the AICc tend to select models with many explanatory variables, the BIC results in more parsimonious models. However, the theoretical bases underlying the two approaches (AIC, BIC) are different, the use of the AIC being primarily for the purpose of prediction, and not for decision-making regarding the statistical significance of the parameters retained in the model.

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