وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

BADJI MOKHTAR-ANNABA UNIVERSITY
UNIVERSITE BADJI MOKHTAR ANNABA

جامعة باجي مختار
- عنابة-

Faculty of Sciences
Department of Mathematics

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THESIS

Presented with a view to obtaining the doctorate degree

New Integral Inequalities and Applications to Boundary Problems

Stream
Applied Mathematics

Speciality
Optimal Control

By
Said Fatma

SUPERVISOR: **Boukerrioua Khaled** **Prof. U.B.M -Annaba**

In front of the jury

PRESIDENT:	**Guezane- Lakoud Assia**	**Prof.**	**U.B.M - Annaba**
EXAMINER:	**Salah Derradji Lylia**	**M.C.A.**	**U.B.M - Annaba**
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Présentée en vue de l'obtention du diplôme de Doctorat

Nouvelles inégalités intégrales et applications aux problèmes aux limites

Filière
Mathématiques Appliquées

Spécialité
Contrôle Optimal
Par
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DIRECTEUR DE THÈSE: Boukerrioua Khaled **Prof. U.B.M -Annaba**

Devant le jury

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Contents

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Résumé

Il est bien connu que la théorie des inégalités intégrales joue un rôle essentiel dans l'étude de l'analyse qualitative et quantitative du comportement des solutions des équations différentielles non linéaires.

L'objectif de cette thèse est d'établir des inégalités intégrales fractionnaires et d 'autres de type Pachpatte-Gamidov aux échelles de temps.

Enfin, nous avons utilisé quelques inégalités intégrales pour étudier la stabilité pratique et la $h-$stabilité des certains systèmes dynamiques non linéaires perturbés aux échelles de temps arbitraires.

Mots-clés: Échelles de temps, Inégalité de Gronwall, Inégalité de Pachpatte-Gamidov , Inégalité de Gronwall- Bellman, Inégalités intégrale fractionnaire , $h-$stabilité, Stabilité pratique, Systèmes perturbés.

Abstract

It is well known that the theory of integral inequalities plays a crucial role in the qualitative and quantitative analysis of the behavior of solutions to nonlinear differential equations.

The objective of this thesis is to establish fractional integral inequalities and other inequalities of the Pachpatte-Gamidov type on time scales.

Finally, we have used certain integral inequalities to study the practical stability and $h-$stability of some perturbed nonlinear dynamical systems on arbitrary time scales.

Keywords : Time scales, Inequality of Gronwall, Inequality of Pachpatte-Gamidov , Inequality of Gronwall- Bellman, Fractional integral inequalitie , Practical stability, $h-$stability, Perturbed systems.

ملخص

من المعروف أن نظرية المتباينات التكاملية تلعب دوراً حيوياً في دراسة سلوك التحليل النوعي والكمي لحلول المعادلات التفاضلية غير الخطية.

الهدف من هذه الاطروحة هو إنشاء متباينات تكاملية كسرية . و اخرى من نوع باشبات- جاميدوف .

وفي الأخير، قمنا باستخدام بعض المتباينات التكاملية لدراسة استقرار الأنظمة الديناميكية غير الخطية المضطربة على سلاسل زمنية.

الكلمات المفتاحية:

السلاسل الزمنية، متباينات جرانوال، متباينات باشبات- جاميدوف، متباينات جرانوال-بالمان، المتباينات التكاملية الكسرية، الاستقرار، الانظمة المضطربة.

Introduction

The Gronwall-Bellman and Bellman-Bihari integral inequalities play significant roles in studying the qualitative and quantitative properties of differential equations [51, 106, 46, 52] . Similarly, discrete Gronwall and Bihari inequalities have been developed for analyzing difference equations[16, 43, 94] and new classes of differential and integral equations have been investigated using Gronwall-Bellman-Pachpatte inequalities [1, 99]. Recently, many researchers have devoted many efforts to investigating weakly singular integral inequalities and their applications [78].Time scale theory, introduced in [62], offers a promising framework that unifies continuous and discrete analyses in a consistent way.

Over the past decades, time scale theory has garnered significant research interest due to its applications in various fields such as economic modeling [4], switched linear circuits [81] , population models [27] , quantum calculus[29] , dynamic programming[102] , neural networks [109], and other scientific domains. The foundational concepts of this mathematical theory were introduced by Hilger[62] . Since its inception, numerous prominent mathematicians have contributed to this field (see, for instance, the monographs [25, 26] and references therein).

Recently, several studies have focused on the asymptotic stability analysis of time-varying linear and nonlinear dynamical systems in both continuous-time and discrete-time settings when nonlinear systems evolve on arbitrary time domains, the stability analysis becomes more challenging. The main cause of instability in dynamic equations on time scales is frequently nonlinearities, thus, there has been a lot of interest in the stability of systems over time scales with linear and nonlinear perturbations [45, 48, 28, 72, 36, 11, 103, 105, 67, 21]. To address these challenges, an integral inequality approach has been developed for time scale dynamic equations in [48, 28, 67, 73, 12, 19, 76] . It is worth emphasizing that such tool can be effectively used for integro-differential dynamics on arbitrary time scales.This is what motivated this investigation which proposes to study the problem of uniform asymptotic stability for some classes of nonlinear integro-differential equations using an integral inequality approach.

The aim of the present work is to provide an overview of classical results on integral inequalities that have appeared in the mathematical literature in recent years, and to establish new nonlinear fractional integral inequalities, new refinements of nonlinear Pachpatte–Gamidov integral inequalities on time scales are also presented, as well as various integral inequalities used to study the practical $h-$stability of certain classes of nonlinear systems.

This thesis is organized into four chapters:

In Chapter 1, we introduce fundamental definitions and basic properties of time scales theory, along with some important fractional integral inequalities.

The aim of the chapter 2 is to study the some properties of solutions to the following initial value problem:

$$\begin{cases} D_r^{\beta}x(t) = f(t, x(t)) \quad t \in (0, \infty), \qquad \beta \in (0,1) \\ \lim_{t\to 0^+} t^{1-\beta}x(t) = x_0, \end{cases} \tag{A}$$

where D_r^{β} is the Riemann–Liouville fractional derivative and the function f satisfies certain inequalities.

In chapter 3, we establish some new fractionnal integral inequalities to study the following general mixed nonlinear integral equation:

$$y^p(t) = x(t) + h(t)\int_{\alpha}^{\eta} F(s, y^q(s))\Delta s + \int_{\alpha}^{\beta} G(s, y^r(s))\Delta s, \tag{B}$$

where $p > 1$ is a constant, y,$x : \mathbb{T} \to \mathbb{R}$ are right-dense continuous functions on $\mathbb{T}$, F :$\mathbb{T} \times\mathbb{T} \times\mathbb{R} \to \mathbb{R}$ is right-dense continuous function on $\mathbb{T} \times\mathbb{T}$ and continuous on $\mathbb{R}$, and G: $\mathbb{T}\times \mathbb{R} \to \mathbb{R}$ is right-dense continuous function on $\mathbb{T}$ and continuous on $\mathbb{R}$.

The chapter 4 treats the concept of practical uniform $h-$stability for the following perturbed dynamic equation :

$$x^{\triangle}(t) = A(t)x + B(t)x + F(t,x). \tag{C}$$

Where $A(.), B(.) \in C_{rd}(\mathbb{T}, M_n(\mathbb{R}))$ and $F : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in time and locally Lipschitz in x, uniformly in t.

The practical uniform h -stability as an extension of practical uniform exponential stability.We give sufficient conditions that guarantee practical uniform $h-$stability for a time perturbed dynamic equation using Gronwall-Bellman inequality.

CHAPTRE 1 Preliminaries

In this chapter, we introduce the concepts necessary for a well understanding of this manuscript, this chapter is organized as follows.The first section includes a brief reminder of the basic elements of time scale theory. Most of these results will be stated without proof. The proofs can be viewed in [25] and [26]. In the second section, we briefly recall certain definitions and properties linked to the theory of fractional calculus.The third section is devoted to the presentation of some fundamental notions and definitions relating to stability. In the last section of this chapter , we present some important lemmas that needed in the sequel.

1.1 Time scale theory

The theory of time scales was introduced by **Stéphan Hilger** [62] in his thesis doctorate in 1988. This theory makes it possible to unite continuous and discrete analysis.

It has experienced considerable growth in recent years to explain several discrete phenomena notably in economics, biology, engineering and information matic, thus the finite difference equation are used extensively to make advance this science. It is very useful for describing seasonal phenomena. For example, it could be for the study of a population of insects which after a certain time disappears, to reappear later after having been for a certain time under larva form.

1.1.1 Definitions

Definition 1.1.1 *A time scale* $\mathbb{T}$ *is a non-empty closed subset of the set real numbers* $\mathbb{R}$*. Jump operators*

Remark 1.1.1 *The topology of* $\mathbb{T}$ *is induced by that of* $\mathbb{R}$*.*

Jump operators

Definition 1.1.2 *Let* T *be a time scale. We define*

i) the forward jump operator: $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\sigma(t) = \inf \{s \in \mathbb{T} \ : s > t\}, \tag{1.1}$$

ii) the backward jump operator: $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}. \tag{1.2}$$

In this definition, we put $sup\ \emptyset = inf\ \mathbb{T}$, (ie, $\rho(t) = t$ if $\mathbb{T}$ has a minimum in t and $inf\ \emptyset = sup\mathbb{T}$, (ie, $\sigma(t) = t$ if $\mathbb{T}$ has a maximum at t, where $\emptyset$ designates the empty set .

1.1.2 Classification of points in a time scale $\mathbb{T}$

Let $\mathbb{T}$ be a time scale, t a point on $\mathbb{T}$.

Definition 1.1.3 *We say that* t *is a dense point to the right of* $\mathbb{T}$*,* $t < sup\mathbb{T}$ *(resp. a dense point to the left of* $\mathbb{T}$*), if* $\sigma(t) = t$ *(resp.* $\rho(t) = t$*). We say that* t *is a dense point to the right of* $\mathbb{T}$*,* $t < \sup \mathbb{T}$ *(resp. a dense point to the left of* $\mathbb{T}$*), if* $\sigma(t) = t$ *(resp.* $\rho(t) = t$*).*

Definition 1.1.4 *We say that t is a dense point if it is simultaneously dense on the right and to the left.*

Definition 1.1.5 *We say that* t *is a scattered point to the right of* $\mathbb{T}$ *(resp. a scattered point to the left of* $\mathbb{T}$*), if* $\sigma(t) > t$ *(resp.* $\rho(t) < t$*).*

Definition 1.1.6 *t is called an isolated point if it is simultaneously dispersed to the right and to the left.*

Definition 1.1.7 *We call the graininess function the function defined by*

$$\mu : \mathbb{T} \to [0, +\infty), \ \mu(t) = \sigma(t) - t. \tag{1.3}$$

[Tab.1.1] The following table summarizes the classification of points in a time scale.

point	The description
dense on the right	$t = \sigma(t)$
dense on the left	$t = \rho(t)$
dense	$\sigma(t) = t = \rho(t)$
scattered to the right	$t < \sigma(t)$
scattered left	$\rho(t) < t$
isolated	$\rho(t) < t < \sigma(t)$

Now, we illustrate the previous definitions with some sets of time scales as well as their characteristics (jump operator, granulation function) indicated in the table below

[Tab.1.2] Some time scales and their characteristics.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{N}^k$	$q^{\mathbb{Z}} \cup \{0\}$	$p^{\mathbb{N}_0} \cup \{0\}, p \in (0,1)$	$\sqrt{2n+1}$
$\sigma(t)$	t	$t+1$	$(1+\sqrt[k]{t})^k$	qt	$\frac{t}{p}$	$\sqrt{t^2+2}$
$\rho(t)$	t	$t-1$	$\left(\sqrt[k]{t}-1\right)^k$	$\frac{t}{q}$	pt	$\sqrt{t^2-2}$
$\mu(t)$	0	1	$(1+\sqrt[k]{t})^k - t$	$(q-1)t$	$t(\frac{1}{p}-1)$	$\sqrt{t^2+2}-t$

1.1.3 Subsets derived from a time scale

We note that from each time scale we can extract the subsets following:

Definition 1.1.8 *Let $\mathbb{T}$ be a time scale, the set $\mathbb{T}^k$ is defined as follows:*

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}\ ] & , \sup \mathbb{T} < \infty \\ \mathbb{T} & , \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1.1.9 *Let $a; b$ two points of $\mathbb{T}$, the time scale interval is defined by :*

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} \ : a \leq t \leq b\}.$$

Remark 1.1.2 *We denote by $[a; b]^k = [a; b) = [a; \rho(b)]$ in the case where b is a point scattered on the left, otherwise $[a; b]^k = [a; b]$.*

1.1.4 Derivation on time scales

In this part, we recall the definition of the Δ-derivative also known as the derivative at meaning of Hilger.

Definition 1.1.10 *Let $f : \mathbb{T} \to \mathbb{R}$ a function and $t \in \mathbb{T}^k$. We will say that f is Δ-differentiable at t if there exists a number $f(t)$ such that for all $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}$ of t where*

$$|[f(\sigma(t)) - f(s)] - f^{\triangle}(t)\,[\sigma(t) - s] \leq \varepsilon\,|\sigma(t) - s| \ \ , \forall \ s \in \mathcal{U}$$

If f is Δ-differentiable at every point $t \in \mathbb{T}^k$, then the function f^{Δ}: $\mathbb{T}^k \to \mathbb{R}$ is called the Δ-derivative of f on $\mathbb{T}^k$. Let $f : \mathbb{T} \to \mathbb{R}$ is function and $t \in \mathbb{T}^k$.

Let us recall some properties of the derived delta which are used in this work.

Theorem 1.1.1 *Let $f : \mathbb{T} \to \mathbb{R}$ a function and let $t \in \mathbb{T}^k$. So we have the following properties:*

1.*If f is Δ-differentiable at t, then f is continuous at t.*

2.*If f is continuous at t and if t is right-scattered, then f is Δ-differentiable at t and*

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \tag{1.4}$$

3.*If t is right dense, then f is Δ-differentiable at t if and only if*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists and is bounded. In this case we have:

$$f^{\triangle}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

4.*If f is $\Delta-$differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\triangle}(t). \tag{1.5}$$

1.1.5 Properties of the Δ-derivative

Theorem 1.1.2 *Let $f; g$:$\mathbb{T} \to \mathbb{R}$ two Δ-differentiable functions in $t \in \mathbb{T}^k$. Then we have the following properties:*

1. $f + g$ is Δ-differentiable in t, moreover

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$$

2. (αf) is Δ-differentiable in t for all $\alpha \in \mathbb{R}$ and so:

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$$

3. fg is Δ-differentiable in t and so:

$$(fg)^{\Delta}(t) = f^{\Delta}(t)\, g(t) + f(\sigma(t))\, g^{\Delta}(t) = f(t)\, g^{\Delta}(t) + f^{\Delta}(t)\, g(\sigma(t))$$

4. If $f(t) f(\sigma(t)) \neq 0$, then $\dfrac{1}{f}$ is Δ-differentiable in t and so:

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t) f(\sigma(t))}.$$

5. If $g(t) g(\sigma(t)) \neq 0$, then $\dfrac{f}{g}$ is Δ-differentiable at t and so:

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t) g(\sigma(t))}.$$

1.1.6 Derivation of composed functions

Theorem 1.1.3 *Let $g : \mathbb{R} \to \mathbb{R}$ a continuous function, $g : \mathbb{T} \to \mathbb{R}$ is delta-differentiable in $\mathbb{T}^k$ and if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then there exists $c \in [t; \sigma(t)]$ satisfying*

$$(f \circ g)^{\Delta}(t) = f^{'}(g(c)) g^{\Delta}(t)$$

Theorem 1.1.4 *Let $f : \mathbb{R} \to \mathbb{R}$, a continuously differentiable function and $g : \mathbb{T} \to \mathbb{R}$, a function differentiable on $\mathbb{T}^k$. Then $f \circ g$ is Δ-differentiable and we have the the following formula:*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f^{'}\left(g(t) + h\mu(t) g^{\Delta}(t) \right) dh \right\} g^{\Delta}(t)$$

1.1.7 Integration

Naturally, calculations on time scales will not be complete without the introduction of Δ-integrability. We then define the functions which are integrable and for this, we present the following definitions:

Definition 1.1.11 *We will say that a function $f : \mathbb{T} \to \mathbb{R}$ is regular if its limit to right exists at every dense point to the right of $\mathbb{T}$ and its limit on the left exists at every point dense to the left of $\mathbb{T}$.*

Definition 1.1.12 *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous in any dense point to the right of $\mathbb{T}$ if its left limit exists and is finite at every dense point to the left of $\mathbb{T}$:*

Remarks 1.1.3 *The set of all rd-continuous functions on $\mathbb{T}$ is denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}; \mathbb{R})$$

Remarks 1.1.4 *The set of all differentiable and rd-continuous functions on $\mathbb{T}$is denoted by:*

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}; \mathbb{R})$$

Definition 1.1.13 *The function F: $\mathbb{T} \to \mathbb{R}$, is called primitive of $f : \mathbb{T} \to \mathbb{R}$, if it verifies $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$*

Definition 1.1.14 *Every rd$-$continuous function $f : \mathbb{T} \to \mathbb{R}$, admits a primitive F: $\mathbb{T} \to \mathbb{R}$, and we note*

$$\int_s^r f(t)\,\Delta t = F(r) - F(s) \quad \text{for all } s, r \in \mathbb{T}$$

Theorem 1.1.5 *If $f \in C_{rd}(\mathbb{T})$ and $t \in \mathbb{T}^k$, we have:*

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = \mu(t) f(t)$$

Theorem 1.1.6 *Let $a; b; c \in \mathbb{T}$; $\lambda \in \mathbb{R}$ and $f; g \in C_{rd}(\mathbb{T}^k)$ than:*

1. $\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t$.
2. $\int_a^b (\lambda f(t))\,\Delta t = \lambda \int_a^b f(t)\,\Delta t$.
3. $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$.
4. $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t.$
5. $\int_a^a f(t)\,\Delta t = 0.$
6. If $|f(t)| \leq g(t)$ on $[a,b]_{\mathbb{T}}$, than $\left|\int_a^b f(t)\,\Delta t\right| \leq \int_a^b g(t)\,\Delta t.$
7. If $f(t) \geq 0$ for $t \in [a,b] \cap \mathbb{T}$, than $\int_a^b f(t)\Delta t \geq 0.$

■ For $\mathbb{T} = \mathbb{R}$, we have:

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt.$$

■ For$\mathbb{T} = \mathbb{Z}$, we have:

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) \text{ if } a < b \\ 0 \text{ if } a = b \\ -\sum_{t=b}^{a-1} f(t) \text{ if } a > b. \end{cases}$$

■ If $[a,b]$ consists of only isolated points, then :

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t \in [a,b[} \mu(t) f(t) \text{ if } a < b \\ 0 \text{ if } a = b \\ \sum_{t \in [b,a[} \mu(t) f(t) \text{ if } a > b. \end{cases}$$

Proposition 1.1.1 *If $a,\ b \in \mathbb{T}$ and $f,\ g \in C^1_{rd}(\mathbb{T}, \mathbb{R})$, then*

$$\begin{aligned} \int_a^b f^\sigma(t)\, g^\Delta(t)\,\Delta t &= [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)\, g(t)\,\Delta t, \\ \int_a^b f(t)\, g^\Delta(t)\,\Delta t &= [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)\, g^\sigma(t)\,\Delta t. \end{aligned}$$

Theorem 1.1.7 *Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ and $L : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$, is continuous at (t,t), for $t \in \mathbb{T}^k$, $t > a$ and $L^{\Delta}(t,.)$ is* rd-*continuous on $[a, \sigma(t)]$, Suppose that for each $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t, independent of $\tau \in [t, \sigma(t)]$, such that:*

$$\left| L(\sigma(t), \tau) - L(s, \tau) - L^{\Delta}(t, \tau)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right|, \quad \forall s \in \mathcal{U},$$

where f^{Δ} denotes the derivative of f with respect to the first variable. Then

$$g(t) = \int_a^t L(t, \tau) \Delta\tau \Rightarrow g^{\Delta}(t) = \int_a^t L^{\Delta}(t, \tau) \Delta\tau + L(\sigma(t), \tau),$$

and

$$h(t) = \int_t^b L(t, \tau) \Delta\tau \Rightarrow h^{\Delta}(t) = \int_t^b L^{\Delta}(t, \tau) \Delta\tau - L(\sigma(t), \tau).$$

1.1.8 The exponential function on time scales

In this subsection, we define an important function on a time scale which generalizes the exponential function on $\mathbb{R}$, $e_p(., t_0)$.

Definition 1.1.15 *Let $h > 0$, we define the Hilger complex numbers by:*

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq \frac{1}{h} \right\}$$

*and **Hilger**'s imaginary axis*

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \operatorname{Im} z < \frac{\pi}{h} \right\}.$$

For $h = 0$, we pose by definition $\mathbb{C}_0 = \mathbb{C}$, $\mathbb{Z}_0 = \mathbb{C}$.

Definition 1.1.16 *We say that the function $p : \mathbb{T} \to \mathbb{R}$ is regressive if:*

$$1 + \mu(t) p(t) \neq 0. \quad \text{for all } t \in \mathbb{T}^k.$$

Definition 1.1.17 *The set of all positive regressive functions is defined by:*

$$\Re^{+} = \left\{ p \in \Re : 1 + \mu(t) p(t) > 0, \text{ for all } t \in \mathbb{T}^k \right\}. \tag{1.6}$$

Proposition 1.1.2 *The set* $\Re$ *provided with the addition* $\oplus$ *defined by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \text{ for all } t \in \mathbb{T}^k, \ p, \ q \in \Re \ .$$

is an abelian group. We call it the regressive group. The conjugate of each element p *of group* $\Re$ *denoted by* $\ominus p$ *is given by:*

$$\ominus p = \frac{-p(t)}{1+\mu(t)p(t)}, \text{ for all } t \in \mathbb{T}^k.$$

Definition 1.1.18 *For* $h \geq 0$*, the cylindrical transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *is defined by:*

$$\xi_h(z) = \begin{cases} \frac{1}{h}\log(1+zh) & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Definition 1.1.19 *We assume that* $p \in \Re$ *and* $t_0 \in \mathbb{T}$ *a fixed point, then the initial value problem*

$$y^{\Delta}(t) = p(t)\, y(t), \ y(t_0) = 1, \tag{1.7}$$

admit a unique solution in $\mathbb{T}$.

Definition 1.1.20 *Let* $p \in \Re$*, the exponential function is defined as the solution of problem* (1.7) *and we have*

$$y(t) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \text{ for } t, t_0 \in \mathbb{T},$$

we often note it by

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \text{ for } t, t_0 \in \mathbb{T},$$

where ξ_μ *is the cylindrical transformation given by definition* 1.1.18

Proposition 1.1.3 *Let* $p, q \in \Re$ *and* $t, t_0, s \in \mathbb{T}$, *Then*

1) $e_0(t, t_0) \equiv 1$ and $e_p(t, t) \equiv 1$,

2) $e_p(\sigma(t), t_0) = (1 + \mu(t)\, p(t))\, e_p(t, t_0)$,

3) $e_p(t, t_0)\, e_p(t_0, s) = e_p(t, s)$,

4) $e_p(t,t_0) = \frac{1}{e_p(t_0,t)}$,

5) If $p \in \Re^+$, $q \in \Re^+$ and $p \leq q$, then $e_p(t,t_0) \leq e_q(t,t_0)$,

6) If $p \geq 0$ then $e_p(.,t_0)$ is an increasing function and $e_p(t,t_0) \geq 1$,

7) If $p \in \Re^+$, then $e_p(t,t_0) > 0$,

8) $e_p(t,t_0)\, e_q(t,t_0) = e_{p\oplus q}(t,t_0)$ or $(p \oplus q)(t) := p(t) + q(t) + \mu(t)\, p(t)\, q(t)$,

9) The function $e_p(.,s)$ is $\Delta-$differentiable in t and $(e_p(.,s))^{\Delta}(t) = p(t)e_p(t,s)$.

Remark 1.1.5 *Let $p \in \Re$, $t, t_0 \in \mathbb{T}$ and $t > t_0$,*

1. *if $\mathbb{T} = \mathbb{R}$, then $e_p(t,t_0) = \exp\left(\int_{t_0}^{t} p(\tau)\, d\tau\right)$,*

2. *if $\mathbb{T} = \mathbb{Z}$, then $e_p(t,t_0) = \prod_{\tau=t_0}^{\tau=t-1} (1 + p(\tau))$,*

1.2 Fractional Calculation Elements

1.2.1 Historical context

The theory of fractional calculus is nearly as old as differential calculus itself, dating back to the era of **Leibniz**, **Gauss**, and **Newton**, who pioneered this branch of mathematics. By the late 19th century, several prominent mathematicians, including **P.S. Laplace** (1812), **J.B. Fourier** (1822), **N.H. Abel** (1823–1826), **J. Liouville** (1832–1873), **B. Riemann** (1847), **A.K. Grünwald** (1867–1872), and **A.V. Letnikov** (1868–1872), made significant contributions to this field. In recent years, the study of fractional-order differential equations has seen substantial development

Many definitions were then given on fractional derivation and integration.

Fractional calculus has several areas of application, for example, in viscoelasticity, control theory, diffusion equation, electricity, biology, electromagnetics [61, 85, 54, 55].

1.2.2 Specific functions for fractional derivation

In this part, we present the **Gamma, Beta** and **Mittag-Leffler** functions. These functions play an important role in the theory of fractional calculus.

The Gamma function

One of the basic functions of fractional calculus is the Euler **Gamma function** $\Gamma(z)$. The Gamma function $\Gamma(z)$ is defined by the following integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.8}$$

with $\Gamma(1) = 1$, $\Gamma(0_+) = +\infty$, $\Gamma(z)$ is a monotonic and strictly decreasing function for $0 < z \leq 1$. An important property of the **Gamma function** $\Gamma(z)$ is the following recurrence relation

$$\Gamma(z+1) = z\Gamma(z),$$

That it can be demonstrated by integration by parts

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt = \left[-e^{-t}t^z\right]_0^\infty + z\int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z).$$

Euler's **Gamma function** generalizes the factorial because

$$\Gamma(n+1) = n!, \forall n \in \mathbb{N}.$$

The Beta function

Definition 1.2.1 *The function $B(p,q)$ is the Beta function (or Eulerian integral of the first kind), defined by:*

$$B(p,q) = \int_0^1 (1-t)^{p-1}\, t^{q-1} dx \quad p > 0,\ q > 0. \tag{1.9}$$

We have an equality expressing the link between the Eulerian integral of the first and second kind:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma\left(p+q\right)},\ Re(p) > 0,\ Re(q) > 0. \tag{1.10}$$

The Mittag-Leffler function

The exponential function, e^z, plays a very important role in the theory of integer differential equations. The generalization of the exponential function to a single parameter was introduced by **G.M. Mittag-Leffler [86]** and [87] denoted by the following function [49] and [50] :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{1.11}$$

The two-parameter Mittag-Leffler function also plays a very important role in the theory of fractional calculus. The latter was introduced by **Agarwal** [5] and it is defined by the following series expansion [49] and [50]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, (\alpha > 0, \beta > 0)\,. \tag{1.12}$$

The Mittag-Leffler function is reduced to simple functions. For example,

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

For differential equations of fractional order, the Mittag-Leffler function plays the same role as the exponential function.

1.2.3 Fractional integral in the sense of Riemann-Liouville

Like the majority of introductory works on fractional calculus, we will follow Riemann's approach to propose a first definition of a fractional integral, the **Riemann-Liouville integral**.

Definition 1.2.2 *Let* $f \in L^1([a,b])$, *the fractional integral in the sense of* ***Riemann-Liouville*** $I_a^\alpha f(x)$ *of order* $\alpha \in \mathbb{C}$ $(\operatorname{Re}(\alpha) > 0)$ *is defined by:*

$$(I_a^{(\alpha)} f) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ (x > a)\,, \tag{1.13}$$

Where Γ *is the Gamma function.*

1.2.4 Fractional derivative in the sense of Riemann-Liouville (R-L).

There are several definitions of fractional derivatives. In this part, we will present the **Riemann-Liouville** derivative, it is the most used.

Definition 1.2.3 *Let f be an integrable function on the interval $[a,b]$, the derivative of non-integer order α (with $n-1<\alpha<n$, $n\neq 0$) in the sense of* (R-L) *defined by:*

$$\begin{aligned} D_a^\alpha f(x) &= D^n I_a^{n-\alpha} f(x) = \frac{d^n}{dx^n}\left[I_a^{n-\alpha} f(x)\right], \\ D_a^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1} f(t)dt. \end{aligned} \tag{1.14}$$

where $n=[\alpha]+1$. In particular, if $\alpha=0$ we will have $D_a^0 f(x)=f(x)$.

Moreover if $0<\alpha<1$, then $n=1$, hence

$$D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha} f(t)dt.$$

1.2.5 Some properties of integration and fractional derivation in the sense of Riemann-Liouville

- For any continuous function f and all $\alpha,\beta>0$, we have

$$I_{a^+}^\alpha(I_{a^+}^\beta f(x)) = I_{a^+}^\beta(I_{a^+}^\alpha f(x)) = I_{a^+}^{\alpha+\beta} f(x).$$

- For all $\alpha>0$ and $f\in C[a,b]$, we have

$$D_{a^+}^\alpha(I_{a^+}^\alpha f(x)) = f(x).$$

1.3 Fundamental notions of stability

In mathematics, the theory of stability deals with the qualitative behaviors of solutions of differential equations and trajectories of dynamic systems under small perturbation on the initial conditions.

The first study of stability at time scales was carried out in 1992 using the Lyapunov method.

Definitions of stability of systems on time scales are obtained by a slight modification of their standard definitions for ordinary differential equations.

Choi [42] and **Hamza** [59], **Dacunha** [45] gave generalized characterizations of the different types of stability (uniform stability, exponential stability, $h-$stability, ...) for solutions of dynamic systems on time scales.

Let us consider the dynamic equation defined on a time scale $\mathbb{T}$ as below

$$x^{\Delta}(t) = f(t, x), \ x(t_0) = x_0 \tag{1.15}$$

where $t \in \mathbb{T}$, $t_0 \in \mathbb{T}_{t_0}^{+}$and $x : \mathbb{T}_{t_0}^{+} \rightarrow \mathbb{R}^n$ is the state vector, $f : \mathbb{T}_{t_0}^{+} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a rd-continuous vector-valued function . Let us also assume that the conditions of existence and uniqueness of solution of the system (1.15) on $\mathbb{T}_{t_0}^{+}$ are satisfied for any initial condition $(t_0, x_0) \in \mathbb{T}_{t_0}^{+}\times \mathbb{R}^n$. Let's designate this solution by $x(t) = x(t; t_0; x_0)$.

where $t \in \mathbb{T}$ and $f : T\times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a continuous

Definition 1.3.1 *The dynamic system* (1.15) *is said to be uniformly stable there exists a finite constant* $\gamma > 0$ *such that for all* $t_0 \in \mathbb{T}$ *and* $x(t_0)$*, any solution* $x(t)$ *satisfies*

$$\|x(t)\| < \gamma \|x(t_0)\|, t \geq t_0. \tag{1.16}$$

Definition 1.3.2 *The dynamic system* (1.15) *is said to be uniformly exponential stable if it exists* $\lambda > 0$ *with* $-\lambda \in \mathcal{R}^{+}$ *and* $\gamma > 1$ *independent of any initial point* t_0*, such such that for all* $t_0 \in \mathbb{T}$ *and* $x(t_0)$*, any solution* $x(t)$ *satisfies:*

$$\|x(t)\| < \gamma \|x(t_0)\| e_{-\lambda}(t, t_0) \ , \ \ forall \ \ t \geq t_0 \tag{1.17}$$

Definition 1.3.3 ([*90*]) *The dynamic system* (1.15) *is said to be globally uniformly h-stable if there exist a non increasing bounded rd-continuous function* $h : \mathbb{T}_{t_0}^{+}\rightarrow \mathbb{R}_{+}^{*}$ *, and a constant* $c \geq 1$*(independent of* t_0*) such that, any solution* $x(t) = x(t, t_0, x_0)$ *of equation* (1.15) *satisfies*

$$\|x(t, t_0, x_0)\| \leq c\|x_0\| h(t) (h(t_0))^{-1}, \ t \in \mathbb{T}_{t_0}^{+}, \tag{1.18}$$

(here $(h(t))^{-1} = \frac{1}{h(t)}$*) and* $\| h \|_{\infty} = \sup_{t\in\mathbb{T}} | h(t) |$.

Definition 1.3.4 ([66]) *The dynamic system* (1.15) *is said to be globally practically uniformly h-stable if there exist* $c \geq 1$ *and* $\rho > 0$*, such that for all* $t \geq t_0$ *and* $x_0 \in \mathbb{R}^n$ *we have*

$$\|x(t)\| \leq \frac{c\,\|x_0\|\,h(t)}{h(t_0)} + \rho. \tag{1.19}$$

The analysis of the stability of non-autonomous linear systems can be completely characterized in terms of the resolvent of the system studied. Indeed, let us consider the problem

$$x^{\Delta}(t) = A(t)x(t),\ x(t_0) = I_n \tag{1.20}$$

where $x, x_0 \in \mathbb{R}^n, t, t_0 \in \mathbb{T}$ and $A : \mathbb{T} \longmapsto \mathrm{R}^{n\times n}$ a regressive $rd-$continuous matrix depends on t.

Definition 1.3.5 *A mapping* $A : \mathbb{T} \to M_n(\mathbb{R})$ *is called regressive if for each* $t \in \mathbb{T}^k$ *the* $n \times n$ *matrix* $I_n + \mu(t)A$ *is invertible, where* I_n *is the identity matrix.*

The class of all regressive and rd-continuous functions A *from* $\mathbb{T}$ *to* $M_n(\mathbb{R})$ *is denoted by* $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

Definition 1.3.6 *For* $t_0 \in \mathbb{T}$ *, the solution of system (1.16) (with* $x(t_0) = I_n$*) is called exponential of the matrix function, denoted by* $\phi_A(t, t_0)$*, where* $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

Accordingly, the matrix function $\phi_A(t, t_0)$ *possesses the following two properties:*

$$\begin{aligned} \phi_A^{\Delta}(t, t_0) &= A(t)\phi_A(t, t_0), \\ \phi_A(t_0, t_0) &= I_n. \end{aligned} \tag{1.21}$$

This matrix function is referred to as the state transition matrix, and our assumption in the nature of $A(t)$ *turns out that the state transition matrix exists and is unique.*

Remark 1.3.1 *If* $A(t) = A$ *where* A *is constant, then* $\phi_A(t, t_0) = e_A(t, t_0)$ *is the exponential of the matrix* A *.*

Theorem 1.3.1 *Suppose* $r, s, t \in \mathbb{T}$*and* $A, B \in C_{rd}\mathcal{R}(\mathbb{T}; M_n(\mathbb{R}))$*, then the transition matrix has the following properties:*

♣ $\phi_A(t,r)\phi_A(r,s) = \phi_A(t,s)$;

♣ $\phi_A(\sigma(t),s) = (I + \mu(t)A(t))\phi_A(t,s)$;

♣ *If* $\mathbb{T} = \mathbb{R}$ *and* A *is constant, then* $\phi_A(t,s) = e_A(t,s) = e^{A(t-s)}$.

♣ *If* $\mathbb{T} = h\mathbb{Z}$, *with* $h > 0$, *and* A *is constant, then* $\phi_A(t,s) = (I + hA)^{\frac{(t-s)}{h}}$.

Uniform stability can be characterized by the transition matrix as follows:

Definition 1.3.7 ([75]) *The system of dynamic equations (1.20) is said to be uniformly stable if and only if there exists a constant* $\gamma > 0$ *such that*

$$\|\phi_A(t,t_0)\| \leq \gamma, \text{ for all } t \geq t_0, t \in \mathbb{T}. \tag{1.22}$$

The notion "uniformly exponentially stable" can be characterized by the transition matrix as follows:

Definition 1.3.8 ([75]) *The system of dynamic equations (1.20) is said to be uniformly exponentially stable if and only if there exist two constants* $\lambda, \gamma > 0$ *with* $-\lambda \in \Re^+$ *such that*

$$\|\phi_A(t,t_0)\| \leq \gamma e_{-\lambda}(t,t_0), \text{ for all } t \geq t_0, t, t_0 \in \mathbb{T}. \tag{1.23}$$

The notion "globally uniformly $h-$stable" can be characterized by the transition matrix as follows:

Definition 1.3.9 *The system (1.20) is globally uniformly* $h-$*stable if and only if there exist* $c \geq 1$ *and a positive continuous bounded function* h *on* $\mathbb{R}_+$, *such that for all* $t \geq t_0$.

$$\|\phi_A(t,t_0)\| \leq \frac{ch(t)}{h(t_0)}, \qquad \forall t \geq t_0. \tag{1.24}$$

1.4 Some important inequalities

In this section, we present a short reminder concerning some classes of functions, as well as inequalities useful in our study.

Let $\beta \in (0,1)$ we define the following function spaces:

- $C_\beta(0,T]$: The space of functions $x:(0,T]\to\mathbb{R}$ such that $x(t)=t^{-\beta}y(t)$ for some $y\in C[0,T]$. This space is equipped with the norm $\| x \|_\beta = sup_{0<t\preceq T} t^\beta \mid x(t) \mid$.
- $C_\beta(0,+\infty)$: The space of functions $x:(0,+\infty)\to\mathbb{R}$ such that $x(t)=t^{-\beta}y(t)$ for some $y\in C[0,+\infty)\}$.
- $L^p_{Loc}[0,+\infty)(p\geq 1)$: The space of all real-valued functions that are Lebesgue integrable on every bounded subinterval of $[0,+\infty)$.

These spaces will be used throughout the chapter 2.

Definition 1.4.1 ([20]) *A function $w:\mathbb{R}_+\to\mathbb{R}_+$ is said to belong to class $\mathfrak{F}$, if it satisfies the following conditions :*

$$\begin{aligned} w(x) &> 0 \text{ is nondecreasing and continuous for } x\geq 0,\\ \frac{1}{a}w(x) &\leq w\left(\frac{x}{a}\right) \quad \text{for } a>0 \ . \end{aligned}$$

For example, if $w(x)=x^p, p\geq 1$, *then* $w(\frac{x}{a})=(\frac{x}{a})^p\geq\frac{x^p}{a}=\frac{w(x)}{a}$ *for* $a\in(0,1]$.

Lemma 1.4.1 ([64]) *Suppose that $a\geq 0$, $p\geq q\geq 0$ and $p\neq 0$, then*

$$a^{\frac{q}{p}}\leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}}a+\frac{p-q}{p}\varepsilon^{\frac{q}{p}},$$

for any $\varepsilon>0$.

Lemma 1.4.2 ([25]) (*Comparison Theorem*) *Assume that $u,b\in\mathcal{C}_{rd}$, and $a\in\mathcal{R}^+$. If*

$$u^\Delta(t)\leq a(t)u(t)+b(t) \ , \text{forall } t\in\mathbb{T}$$

then

$$u(t)\leq u(t_0)e_a(t,t_0)+\int_{t_0}^t e_a(t,\sigma(\tau))b(\tau)\Delta\tau, \text{ forall } t\in\mathbb{T}.$$

Lemma 1.4.3 ([17]) *Let $t_0 \in T, \phi, \psi, \varphi, \alpha \in C_{rd}(T, \mathbb{R}_+)$ which satisfy*

$$\phi(t) \leq \varphi(t) + \psi(t) \int_{t_0}^{t} \alpha(s)\phi(s) \triangle s.$$

Then

$$\phi(t) \leq \varphi(t) + \psi(t) \int_{t_0}^{t} \alpha(s)\varphi(t) \exp\left(\int_{\sigma(s)}^{t} \alpha(\tau)\psi(\tau) \triangle \tau\right) \triangle s \quad , \qquad t \in T_{t_0}^{+}$$

Lemma 1.4.4 ([17]) *Suppose that $x, f, g, d, m \in C_{rd}(T, \mathbb{R}_+)$. If*

$$x(t) \leq f(t) + g(t) \int_{t_0}^{t} [d(s)x(s) + m(s)] \triangle s \qquad \forall t \in T_{t_0}^{+}$$

Then

$$x(t) \leq f(t) + g(t) \int_{t_0}^{t} [d(s)f(s) + m(s)] \exp\left(\int_{\sigma(s)}^{t} d(\tau)g(\tau) \triangle \tau\right) \triangle s \qquad t \in T_{t_0}^{+}$$

Theorem 1.4.1 ([6]) *Let E be a Banach space, C a closed, convex subset of E and $0 \in C$. Let $F : C \to C$ be a continuous and completely continuous map,and let the set $\{x \in E : x = \lambda F x \ \ for\ some\ \lambda \in (0,1)\}$ be bounded. Then F has at least one fixed point in E.*

Lemma 1.4.5 ([8, 101]) . *Suppose $\rho \in L^p[0,1]$.Then*

$$\int_{0}^{t} (t-s)^{\beta-1}\rho(s)ds,$$

is continuous on $[0,1]$, where $\beta \in (0,1)$ and $q > \frac{1}{\beta}$.

Lemma 1.4.6 ([83]) *Let α, β, λ, and p be positive constants. Then*

$$\int_{0}^{t} (t^{\alpha} - s^{\alpha})^{p(\beta-1)} s^{p(\lambda-1)} ds = \frac{t^{\theta}}{\alpha} \beta \left[\frac{p(\lambda-1)+1}{\alpha},\ p(\beta-1)+1\right] \quad , \qquad t \in \mathbb{R}_+ \tag{1.25}$$

where

$$B\left[\zeta, \eta\right] = \int_{0}^{1} s^{\zeta-1}(1-s)^{\eta-1} ds \quad (\Re e\ \zeta > 0, \Re e\ \eta > 0), \tag{1.26}$$

is the well-known beta function and

$$\theta = p\left[\alpha(\beta-1) + \lambda - 1\right] + 1 \geq 0. \tag{1.27}$$

CHAPTRE 2

Generalized fractionnal integral inequalities and Applications

2.1 Introduction

The study of integral inequalities has a great importance in the theory of differential equations and applied sciences. In the last two decades, many researchers in mathematics have invested in the development of this theory. Fractional inequalities are also quite important, whose applications are very numerous, particularly in the theory of fractional differential equations and in probability theory. Several authors are interested in the study of integral inequalities, see [10].

In the first part of this chapter, we cite some well-known integral inequalities of the Gronwall-Bellman type.

In the second part we establish new extensions concerning fractional integral inequalities.

Finally, in the third part, we give new generalizations of some results cited in the second part and we conclude this chapter with illustrative examples of these new generalizations.

2.2 Some well-known integral inequalities of the Gronwall type

Since 1919 the Swede **Thomas Hakon Gronwall** has become known for his remarkable differential inequality, sometimes called Gronwall's lemma. Later on, it was applied to the theory of differential equations.

Lemma 2.2.1 *([57]) Let $u(t)$ be a continuous function defined on the interval $[t_0; t_1]$, a and b positive constants. If the inequality*

$$u(t) \leq a + b \int_{t_0}^{t} u(s)ds,$$

is satisfied, then we have

$$u(t) \leq ae^{b(t-t_0)}, [t_0; t_1]$$

In 1943, Bellman generalized Gronwall's result in the case where b is a function which depends on the variable t, its result was the following:

Theorem 2.2.1 ([14]) *Let u and g be two continuous and positive functions on $I = [\alpha, \beta] \subset \mathbb{R}$ and $c \geq 0$. If the inequality l'inégalité*

$$u(t) \leq c + \int_{\alpha}^{t} g(s)u(s)ds,$$

is satisfied, then

$$u(t) \leq c \exp\left(\int_{\alpha}^{t} g(s)ds\right), \; t \in I.$$

In 1956, Bihari proved an inequality even more general than those of **Gronwall** and **Bellman**, whose statement is as follows:

Theorem 2.2.2 ([24]) *Let u, f be two positive continuous functions on $[0, +\infty[$, w an increasing and continuous function on $[0, +\infty[$, satisfied $w(x) > 0$ for all $x > 0$ and c a positive constant. If the inequality*

$$u(t) \leq c + \int_{t_0}^{t} f(s)w(u(s))\, ds,$$

is fulfilled for every $t \geq 0$, *then*

$$u(t) \leq G^{-1}\left(G(c) + \int_{t_0}^{t} f(s)ds\right) \quad \text{for all } 0 \leq t \leq T,$$

where G *is solution of the following integral equation*

$$G(t) = \int_{t_0}^{t} \frac{1}{w(s)} ds, \; t > t_0 > 0,$$

G^{-1} *is the inverse function of* G, T *is chosen such that*

$$\left\{ G(c) + \int_{t_0}^{t} f(s)ds \right\} \in DomG^{-1} \text{ for all } 0 \leq t \leq T.$$

In 1958, **Bellman** established another variation of Theorem 2.2.1 as follows:

Theorem 2.2.3 ([15]) *Let* u *and* g *be two continuous and positive functions on* $I = [\alpha, \beta]$ *and let* $n(t)$ *be a continuous, positive and non-decreasing function defined on* I . *If the inequality*

$$u(t) \leq n(t) + \int_{\alpha}^{t} g(s)u(s)ds, \; t \in I,$$

is satisfied, then

$$u(t) \leq n(t) \exp(\int_{\alpha}^{t} g(s)ds), \; t \in I.$$

In 1967, **Chu and Metcalf** established a variant of the **Gronwall-Bellman** inequality in the case where the function g will depend on the parameter t (i.e $g(s) = k(t, s)$), the statement of which is following:

Theorem 2.2.4 ([43]) *Let* u *and* f *be two continuous and positive functions on* $I = [\alpha, \beta]$ *and* $k(t, s)$ *a continuous and positive function on the triangle* $\Delta : \alpha \leq s \leq t \leq \beta$. *If the inequality*

$$u(t) \leq f(t) + \int_{\alpha}^{t} k(t, s)u(s)ds, \; t \in I,$$

then,

$$u(t) \leq f(t) + \int_{\alpha}^{t} H(t,s)f(s)ds, \ t \in I,$$

where

$$H(t,s) = \sum_{t=1}^{\infty} k_i(t,s), \ (t,s) \in \Delta,$$

is the resolving kernel

In 1969, **Gollwitzer** proved an inequality even more general than that of Bellman given by the following theorem:

Theorem 2.2.5 ([56]) *Let u, f, g and h be continuous and positive functions on $I = [\alpha, \beta]$. If the inequality*

$$u(t) \leq f(t) + g(t) \int_{\alpha}^{t} h(s)u(s)ds,$$

is satisfied, then

$$u(t) \leq f(t) + g(t) \int_{\alpha}^{t} h(s)f(s)\exp(\int_{s}^{t} h(\sigma)g(\sigma)d\sigma)ds, t \in I.$$

I. Györi established in 1971 a variant of Theorem 2.2.2 as follows:

Theorem 2.2.6 ([58]) *Let u and β be two continuous and positive functions on $I = [t_0, \infty)$, and let f, g and α be differentiable functions with f positive, g positive and increasing and α positive and decreasing. Suppose that*

$$u(t) \leq f(t) + \alpha(t) \int_{t_0}^{t} \beta(s)g\left(u(s)\right)ds,$$

If

$$f'\left(t\right)\left\{\frac{1}{g\left(\eta\left(t\right)\right)} - 1\right\} \leq 0, \ \text{on } I,$$

for any continuous and positive function $\eta\left(t\right)$, then

$$u(t) \leq G^{-1}\left\{G\left(f(t_0)\right) + \int_{t_0}^{t} \left[\alpha(s)\beta(s) + f'\left(s\right)\right]ds\right\},$$

where G is solution of the following integral equation

$$G(t) = \int_{t_0}^{t} \frac{1}{g(s)} ds, \quad t > t_0 > 0,$$

G^{-1} is the inverse function of G,

$$\left\{ G\left(f(t_0)\right) + \int_{t_0}^{t} \left[\alpha(s)\beta(s) + f'(s)\right] ds \right\} \in DomG^{-1}.$$

In 1987, Norbury and Stuart established another variation of Theorem 2.2.4.

Theorem 2.2.7 ([88]) *Let $u(t)$ and $k(t,s)$ be defined as in Theorem 2.2.4 such that $k(t,s)$ is increasing with respect to t for all $s \in I$.*

i) If the inequality

$$u(t) \leq c + \int_{\alpha}^{t} k(t,s)u(s)ds,$$

is satisfied for any constant $c \geq 0$, then

$$u(t) \leq c \exp\left(\int_{\alpha}^{t} k(t,s)ds \right), \ t \in I.$$

ii) Let $n(t)$ be a continuous, positive and increasing function for all $t \in I$. If the inequality

$$u(t) \leq n(t) + \int_{\alpha}^{t} k(t,s)u(s)ds,$$

is satisfied, then

$$u(t) \leq n(t) \exp\left(\int_{\alpha}^{t} k(t,s)ds \right), \ t \in I.$$

2.3 Some extensions of Henry- type integral inequalities

In 1981, **Henry** obtained the following result about weakly singular Gronwall type inequalitys.

Theorem 2.3.1 ([60]) *Suppose $\beta > 0, \gamma > 0$, $\beta + \gamma > 1$ and $\lambda \geq 0$, $\kappa \geq 0$, x is nonnegative and $t^{\gamma-1}x(t)$ is locally integrable on $0 \leq t < T$, and u satisfies*

$$x(t) \leq \lambda + \kappa \int_0^t (t-s)^{\beta-1} s^{\gamma-1} x(s) ds, \quad a.e. t \in [0, T).$$

Then

$$x(t) \leq \lambda E_{\beta,\gamma}(\kappa \Gamma(\beta)^{\frac{1}{\beta+\gamma-1}} t),$$

where $E_{\beta,\gamma}(z)$ is given by an infinite series related to the two-parameter Mittag-Leffler function.

Lemma 2.3.1 ([*60*]) *Suppose $\kappa > 0$, $\beta > 0$ and is nonnegative functions locally integrable on $0 \leq t < T$, (some $T \leq +\infty$), and suppose $x(t)$ is nonnegative and locally integrable on*

$0 \leq t < T$ with

$$x(t) \leq \lambda(t) + \kappa \int_0^t (t-s)^{\beta-1} x(s) ds$$

on this interval;

then

$$x(t) \leq \lambda(t) + \theta \int_0^t E'_\beta \left(\theta(t-s)\right) \lambda(s) ds, 0 \leq t < T,$$

where

$$\eta = (\kappa\Gamma(\beta))^{\frac{1}{\beta}}, \ E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}, \ E'_\beta(z) = \frac{d}{dz} E_\beta(z),$$

$$E'_\beta(z) \simeq \frac{z^{\beta-1}}{\Gamma(\beta)} \ as \ z \to 0^+, E'_\beta(z) \simeq \frac{1}{\beta} e^z \ as \ z \to +\infty \ (\ and \ E_\beta(z) \simeq \frac{1}{\beta} e^z as \ z \to +\infty)$$

If $\lambda(t) \equiv \lambda$, *constant, then*

$$x(t) \leq \lambda E_\beta(\eta t)$$

In 1964, Willett established a different form of the Gronwall-Bellman inequality given by the following lemma:

Lemma 2.3.2 ([*104*]) *Let* $1 \leq p < \infty$, $\Xi(t)$ *and* $\Lambda(t)$ *be continuous and nonnegative functions on* $[0, \infty)$, $j(t)$ *be a nonnegative and continuous function on* $(0, +\infty)$ *and* $j(t) \in L^1_{Loc}[0, +\infty)$.*Suppose* $\Theta(t)$ *is a nonnegative continuous function on* $[0, +\infty)$ *with*

$$\Theta(t) \leq \Xi(t) + \Lambda(t)(\int_0^t j(s)\Theta^p(s)ds)^{\frac{1}{P}}, \quad t \in [0, \infty). \tag{2.1}$$

then

$$\Theta(t) \leq \Xi(t) + \Lambda(t)\frac{(\int_0^t j(s)e(s)\Xi^p(s)ds)^{\frac{1}{P}}}{1 - [1 - e(t)]^{\frac{1}{p}}},$$

where

$$e(t) = exp(-\int_0^t j(s)\Lambda^p(s)ds).$$

Medved' establishes a generalization of lemma 2.3.1 as follows:

Theorem 2.3.2 ([**?**]) *Let* $a(t)$ *be a nondecreasing, nonnegative* C^1*-function on* $\langle 0, T)$, $L(t)$ *be a continuous, nonnegative function on* $\langle 0, T)$, *and* $\Omega(t)$ *be a continuous, nonnegative function on* $\langle 0, T)$. *with*

$$\Omega(t) \leq \kappa(t) + \int_0^t (t-s)^{\beta-1} L(s)\Omega(s)ds$$

where $\beta > 0$. *Then the following assertions hold:*

If $\beta > \frac{1}{2}$ *then*

$$\Omega(t) \leq \sqrt{2}\kappa(t)\exp\left[\frac{2\Gamma(2\beta-1)}{4^\beta}\int_0^t L(s)^2 ds + t\right], \qquad t \in \langle 0, T)$$

If $\beta = \dfrac{1}{(z+1)}$. *for some* $z \geq 1$, *then*

$$\Omega(t) \leq \left(2^{q-1}\right)^{\frac{1}{q}}\kappa(t)\exp\left[\frac{2^{q-1}}{q}D_z^q\int_0^t L(s)^q ds + t\right], \qquad t \in \langle 0, T)$$

where $D_z = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{\frac{1}{p}}$, $q = z + 2$.

Another result proven by **Medved'**is the following:

Theorem 2.3.3 ([83]) *Suppose $\kappa(t)$, $r(t)$. are nonnegative, integrable functions on $\langle 0, T)$. $(0 < T \leq \infty)$. and $L(t)$, $u(t)$ are integrable, nonnegative functions on $\langle 0, T)$ with*

$$\Omega(t) \leq \kappa(t) + R(t) \int_0^t (t-s)^{\beta-1} L(s)\Omega(s)ds, \quad t \in \langle 0, T)$$

Then the following assertions hold:

If $\beta > \frac{1}{2}$, then

$$\Omega(t) \leq e^t \Upsilon(t)^{\frac{1}{2}}, \quad t \in \langle 0, T)$$

where

$$\Upsilon(t) = 2\kappa(t)^2 + 2DR(t)^2 \int_0^t \kappa(s)^2 L(s)^2 \exp\left[D\int_0^t R(r)^2 L(r)^2 dr\right] ds, \qquad D = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}}$$

If $\beta = \dfrac{1}{(z+1)}$ for some $z \geq 1$, then

$$\Omega(t) \leq e^t F(t)^{\frac{1}{2}}, \quad t \in \langle 0, T)$$

where

$$F(t) = 2^{q-1}\kappa(t)^q + 2^{q-1}D_z^q R(t)^q \int_0^t \kappa(s)^q L(s)^q \exp\left[2^{q-1}D_z^q \int_s^t R(r)L(r)^q dr\right] ds.$$

$q = z + 2$, and $D_z = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{\frac{1}{p}}$.

T. Zhu, also established a new generalization of Lamma 2.3.1 as follows:

Theorem 2.3.4 ([107]) *Let $0 < T \leq \infty$, $\beta \in (0,1)$, $z(t)$ and $v(t)$ are continuous, nonnegative functions on $[0, T)$, and $w(t)$ be a continuous, nonnegative function on $[0, T)$ with*

$$w(t) \leq z(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s)w(s)ds$$

Then

$$w(t) \leq \left(Q(t) + \int_0^t M(s)Q(t)\exp\left(\int_s^t M(\tau)d\tau\right) ds\right)^\alpha, t \in [0, T)$$

If $z(t)$ is nondecreasing on $[0, T)$, then the inequality (2.4) is reduced to

$$w(t) \leq \left(Q(t)\exp\left(\int_0^t M(s)ds\right)\right)^\alpha$$

If $z(t) \equiv 0$ *on* $[0, T)$, *then*

$$w(t) \equiv 0$$

where $Q(t) = 2^{\frac{1}{\alpha}-1} z^{\frac{1}{\alpha}}(t)$, $M(t) = \frac{2^{\frac{1}{\alpha}-1}}{\Gamma^{\frac{1}{\alpha}}(\beta)} \left(\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\right)^{\frac{1-\alpha}{\alpha}} t^{\frac{\beta-\alpha}{\alpha}} v^{\frac{1}{\alpha}}(t)$ *and* $0 < \alpha < \beta < 1$.

Theorem 2.3.5 ([*107*]) *Let* $0 < T \leq \infty$, $\beta > 0$, $z(t)$, $\vartheta(t)$ *and* $v(t)$ *are continuous, nonnegative functions on* $[0, T)$, *and* $w(t)$ *be a continuous, nonnegative function on* $[0, T)$ *with*

$$w(t) \leq z(t) + \frac{\vartheta(t)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) w(s) ds$$

Then

$$w(t) \leq \left(Q(t) + W(t) \int_0^t M(s) Q(t) \exp\left(\int_s^t M(\tau) W(\tau) d\tau\right) ds\right)^{\frac{1}{p}}$$

where $Q(t) = 2^{p-1} z^p(t)$, $W(t) = 2^{p-1}\left(\frac{\vartheta(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} t^{\beta-1+\frac{1}{q}}\right)^p$, $M(t) = v^p(t)$ *and* $p, q \in (0, \infty)$ *such that* $\frac{1}{q} + \beta > 1$ *and* $\frac{1}{q} + \frac{1}{p} = 1$.

In 2020, **Zhu** established the weakly singular integral inequalities of Gronwall-Bellman type, cited by the following theorems:

Theorem 2.3.6 ([*108*]) *Let* $\beta \in (0, 1)$ *and* $\gamma \geq 0$, $z(t)$ *and* $\vartheta(t)$ *be nonnegative and continuous functions on* $[0, +\infty)$, $v(t)$ *be a nonnegative and continuous function on* $(0, +\infty)$ *and* $t^{-\gamma} v(t) \in L^q_{Loc}[0, +\infty)(q > \frac{1}{\beta})$, *and* $w(t)$ *be a continuous, nonnegative function on* $[0, +\infty)$ *with*

$$w(t) \leq z(t) + \vartheta(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} v(s) w(s) ds, \quad t \in [0, \infty),$$

Then

$$w(t) \leq (Q(t) + W(t) \int_0^t M(s) Q(s) \exp(\int_s^t M(\tau) W(\tau) d\tau) ds)^{\frac{1}{q}}, \quad t \in [0, +\infty),$$

where $Q(t) = 2^{q-1} z^q(t)$, $W(t) = \frac{2^{q-1} \vartheta^q(t) t^{q\beta-q+\frac{p}{q}}}{(p\beta--p+1)^{\frac{q}{p}}}$, $M(t) = t^{-q\gamma} v^q(t)$ *and* $p \in (1, +\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.3.7 ([*108*]) *Let* $\beta \in (0,1)$, $z(t)$ *be a nonnegative and continuous function on* $[0,+\infty)$, $v(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $v(t) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$, *and* $w(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$ *with*

$$w(t) \leq z(t) + t^{1-\beta} \int_0^t (t-s)^{\beta-1} s^{\beta-1} v(s) w(s) ds,$$

then

$$w(t) \leq z(t) + \vartheta(t)(Q(t) \exp(\int_0^t M(s) ds))^{\frac{1}{q}}, \quad t \in [0,\infty).$$

where $\vartheta(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $Q(t) = \int_0^t 2^{q-1} v^q(s) z^q(s) ds$, $M(t) = 2^{q-1} v^q(t) \vartheta^q(t)$ *and* $p \in (1,+\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Zhu studied inequality (2.1) by a new method in the following Lemma:

Lemma 2.3.3 ([*108*]) *Let* $1 \leq p < \infty$, $z(t)$ *and* $\vartheta(t)$ *be continuous and nonnegative functions on* $[0,\infty)$, *nonnegative function* $v(t) \in L^p_{Loc}[0,+\infty)$,*and* $w(t)$ *be a continuous and nonnegative function with*

$$w(t) \leq z(t) + \vartheta(t)(\int_0^t v^p(s) w^p(s) ds)^{\frac{1}{p}}, \quad t \in [0,\infty).$$

Then

$$w(t) \leq z(t) + \vartheta(t)(Q(t) \exp \int_0^t M(s) ds)^{\frac{1}{P}}, \quad t \in [0,\infty),$$

where

$$\begin{aligned} Q(t) &= \int_0^t 2^{p-1} v^p(s) z^p(s) ds; \\ M(t) &= 2^{p-1} v^p(t) \vartheta^p(t). \end{aligned}$$

2.4 New generalizations

2.4.1 Fractional integral inequalities

In this section, we will state and prove some new fractional integral inequalities of the Gronwall-Bellman type, which are generalizations of the results obtained by T. Zhu [108]

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Theorem 2.4.1 *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$ with $b(t)$ is nondecreasing, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$. Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a continuous function such that*

$$0 \leq f(t,x) - f(t,y) \leq L(t,y)(x-y), \quad x \geq y \geq 0, \tag{2.2}$$

for $t \in \mathbb{R}_+$ and $x \geq y \geq 0$, where $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous function with $t^{-\gamma}$ $l(t)L(t,a(t)) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$. If

$$u(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds, \quad t \in [0,\infty), \tag{2.3}$$

then

$$u(t) \leq a(t)+b(t)(A(t)+B(t)\int_0^t R(s)A(s)\exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t \in [0,+\infty), \tag{2.4}$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta--p+1)^{\frac{q}{p}}}t^{\beta-1+\frac{1}{q}}$, $R(t) = t^{-q\gamma}r^q(t), p \in (1,+\infty)$ such that $\frac{1}{p}+\frac{1}{q} = 1$.
And

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,a(s))ds \ , \ r(t) = l(t)L(t,a(t)) \ . \tag{2.5}$$

Proof. Let

$$z(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds, \tag{2.6}$$

then, we have $z(0) = 0$ and

$$u(t) \leq a(t) + b(t)z(t). \tag{2.7}$$

So it follows that

$$z(t) \leq \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,a(s)+b(s)z(s))ds, \tag{2.8}$$

from (2.2), we have

$$f(t,a(t)+b(t)z(t)) \leq L(t,a(t))b(t)z(t) + f(t,a(t)), \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$z(t) \leq \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s)[L(s,a(s))b(s)z(s) + f(s,a(s))]ds. \tag{2.10}$$

The inequality (2.10) can be reformulated as

$$z(t) \leq \widetilde{a}(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} r(s) z(s) ds, \tag{2.11}$$

where $\widetilde{a}$ and r are defined as in (2.5).

By Theorem 2.3.6,and using (2.7) we obtain the inequality (2.4). ■

Remark 2.4.1 *Assume* $f(t,u(t)) = u(t)$,*Theorem* 2.4.1 *implies Theorem* 3.1 *in* [*108*].

Corollary 2.4.1 *Let* $\beta \in (0,1)$ *and* $\gamma \geq 0$, $a(t)$ *and* $b(t)$ *be nonnegative and continuous functions on* $[0,+\infty)$*with* $b(t)$ *is nondecreasing,* $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$.*Suppose* $g\colon \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *is a differentiable increasing function on* $]0,+\infty[$ *with continuous nonincreasing first derivative* $g^{'}$ *on* $]0,+\infty[$ *with* $t^{-\gamma}l(t)g'(a(t)) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$.*If*

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(u(s)) ds, \quad t \in [0,\infty),$$

then

$$u(t) \leq a(t) + b(t)(A(t) + B(t) \int_0^t R(s)A(s) \exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t \in [0,+\infty),$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta--p+1)^{\frac{q}{p}}} t^{\beta-1+\frac{1}{q}}$, $R(t) = t^{-q\gamma}r^q(t)$ *and* $p \in (1,+\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.
And

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(a(s)) ds \ , \ r(t) = \ l(t)g'(a(t)) \ .$$

Proof. Applying the mean value Theorem for the function g, then for every $x \geq y > 0$, there exists $c \in \,]\, y, x[$such that

$$g(x) - g(y) = g^{'}(c)(x-y) \leq g^{'}(0)(x-y),$$

The rest of proof is essentially identical to the proof of Theorem 2.4.1. ■

Corollary 2.4.2 *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$ with $b(t)$ is nondecreasing, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$. Suppose $\frac{t^{-\gamma}l(t)}{1+a(t)} \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$. If*

$$u(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)\ln(u(s)+1)ds, \quad t \in [0,\infty),$$

then

$$u(t) \leq a(t) + b(t)(A(t) + B(t)\int_0^t R(s)A(s)\exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t \in [0,+\infty),$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta--p+1)^{\frac{q}{p}}}t^{\beta-1+\frac{1}{q}}$, $R(t) = t^{-q\gamma}r^q(t)$ *and* $p \in (1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$.
And

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)\ln(1+a(t))ds \ , \ r(t) = \frac{b(t)\ l(t)}{\ln(a(t)+1} \ .$$

Theorem 2.4.2 *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$. Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a continuous function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0) = 0$ such that*

$$0 \leq f(t,x) - f(t,y) \leq L(t,y)\phi^{-1}(x-y), \tag{2.12}$$

for $t \in \mathbb{R}_+$and $x \geq y \geq 0$, where $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous function and ϕ^{-1} is the inverse function of ϕ and

$$\phi^{-1}(y_1.y_2) \leq \phi^{-1}(y_1)\phi^{-1}(y_2), \tag{2.13}$$

for x, $y \in \mathbb{R}_+$. If $t^{-\gamma}l(t)\ L(t,a(t)) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$ and

$$u(t) \leq a(t) + b(t)\phi(\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds), \quad t \in [0,+\infty), \tag{2.14}$$

then

$$u(t) \leq a(t) + b(t)\phi(A(t) + B(t)\int_0^t K(s)A(s)\exp(\int_s^t K(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, \quad t \in [0, +\infty), \tag{2.15}$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}\phi^{-1}(b(t))^q}}{(p\beta--p+1)^{\frac{q}{p}}}t^{\beta-1+\frac{1}{q}}$, $K(t) = t^{-q\gamma}m^q(t)$ *and* $p \in (1, +\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

And

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s, a(s))ds \ , \ m(t) = \ l(t)L(t, a(t)) \ . \tag{2.16}$$

Proof. Let

$$z(t) = \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s, u(s))ds. \tag{2.17}$$

Then $z(0) = 0$, and (2.14) can be written as

$$u(t) \leq a(t) + b(t)\phi(z(t)). \tag{2.18}$$

It follows that

$$z(t) \leq \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)f(s, a(s) + b(s)\phi(z(s))ds, \tag{2.19}$$

from (2.12) and (2.13), we observe that

$$\begin{aligned} f(t, a(t) + b(t)\phi(z(t))) &\leq L(t, a(t))\phi^{-1}(b(t)\phi(z(t))) + f(t, a(t)) \\ &\leq L(t, a(t))\phi^{-1}(b(t))z(t) + f(t, a(t)). \end{aligned} \tag{2.20}$$

Using (2.19) and (2.20), we obtain

$$z(t) \leq \int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)[L(s, a(s))\phi^{-1}(b(s))z(s) + f(s, a(s))]ds \ . \tag{2.21}$$

The inequality (2.21) can be reformulated as

$$z(t) \leq \widetilde{a}(t) + \phi^{-1}(b(t))\int_0^t (t-s)^{\beta-1}s^{-\gamma}m(s)z(s)ds, \quad t \in [0, \infty), \tag{2.22}$$

where $\widetilde{a}$ and m are defined as in (2.16).

Applying Theorem 2.3.6 to (2.22), and using(2.18), we can get the desired inequality (2.15). ■

Remark 2.4.2 *Assume* $f(t,u(t)) = u(t)$ *and* $\phi(x) = x$,*Theorem* 2.4.2 *implies Theorem* 3.1 *in [108].*

Corollary 2.4.3 *Let* $\beta \in (0,1)$, $a(t)$ *be a nonnegative and continuous function on* $[0,+\infty)$, $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$. *Let* $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be a continuous function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous and strictly increasing function with* $\phi(0) = 0$ *satisfy* $(2.12)-(2.13)$ *and* $u(t)$ *be a continuous, nonnegative function on* $[0,+\infty)$. *Suppose* $t^{\beta-1}l(t)L(t,a(t)) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$.*If*

$$u(t) \leq a(t) + t^{1-\beta}\phi(\int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)f(s,u(s))ds). \quad (2.23)$$

Then

$$u(t) \leq a(t)+t^{1-\beta}\phi(A(t)+B(t)\int_0^t K(s)A(s)\exp(\int_s^t K(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}, t \in [0,\infty), \quad (2.24)$$

where $A(t) = 2^{q-1}\tilde{a}^q(t)$, $B(t) = \frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}(\phi^{-1}(t^{1-\beta}))^q}{(p\beta--p+1)^{\frac{q}{p}}}t^{\beta-1+\frac{1}{q}}$, $K(t) = t^{-q\gamma}m^q(t)$ *and* $p \in (1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q} = 1$.
And

$$\tilde{a}(t) = \int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)f(s,a(s))ds \ , \ n(t) = l(t)L(t,a(t)) \ . \quad (2.25)$$

Theorem 2.4.3 *Let* $1 \leq p \leq q < \infty$, $a(t)$ *and* $b(t)$ *be continuous and nonnegative functions on* $[0,\infty)$, $l(t)$ *be a nonnegative and continuous function on* $[0,+\infty)$.*Let* $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be a continuous function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous and strictly increasing function with* $\phi(0) = 0$ *satisfies* $(2.12)-(2.13)$.*Suppose* $l(t)L(t,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}}) \in L^p_{Loc}[0,+\infty)$,*and* $u(t)$ *be a continuous and nonnegative function .If*

$$u^q(t) \leq a(t) + b(t)\phi(\int_0^t l(s)f(s,u^p(s))ds), \quad t \in [0,\infty). \quad (2.26)$$

Then

$$u(t) \leq (a(t) + b(t)\phi(\widetilde{a}(t) + \frac{\int_0^t k(s)e(s)\widetilde{a}(s)ds}{1-[1-e(t)]})^{\frac{1}{q}}, \quad t \in [0,\infty). \quad (2.27)$$

where $\widetilde{a}(t) = \int_0^t l(s)f(s,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(s)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}})ds$, $k(t) = l(t)L(t,\frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t)+\frac{q-p}{q}\varepsilon^{\frac{p}{q}})$ *and* $e(t) = exp(-\int_0^t k(s)ds)$.

Proof. Let
$$z(t) = \int_0^t l(s)f(s, u^p(s)ds. \tag{2.28}$$
Then $z(0) = 0$ and (2.26) can be written as
$$u^p(t) \leq (a(t) + b(t)\phi(z(t)))^{\frac{p}{q}}. \tag{2.29}$$

So it follows that
$$z(t) \leq \int_0^t l(s)f(s, (a(s) + b(s)\phi(z(s)))^{\frac{p}{q}})ds \ , \tag{2.30}$$

from Lamma (1.4.1), (2.12) and (2.13), we have
$$\begin{aligned} f(t, (a(t) + b(t)\phi(z(t)))^{\frac{p}{q}}) &\leq L(t, \tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t) + \tfrac{q-p}{q}\varepsilon^{\frac{p}{q}})\phi^{-1}(\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t)\phi(z(t))) \\ &+ f(t, \tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t) + \tfrac{q-p}{q}\varepsilon^{\frac{p}{q}}). \\ &\leq L(t, \tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t) + \tfrac{q-p}{q}\varepsilon^{\frac{p}{q}})\phi^{-1}(\tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t))z(t) \\ &+ f(t, \tfrac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t) + \tfrac{q-p}{q}\varepsilon^{\frac{p}{q}}). \end{aligned} \tag{2.31}$$
Using (2.31) and (2.30), we obtain
$$z(t) \leq \int_0^t l^p(s)[L(t, \frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t) + \frac{q-p}{q}\varepsilon^{\frac{p}{q}})z(t) + f(t, \frac{p}{q}\varepsilon^{\frac{p-q}{q}}a(t) + \frac{q-p}{q}\varepsilon^{\frac{p}{q}})]ds. \tag{2.32}$$
The inequality (2.32) can be restated as
$$z(t) \leq \widetilde{a}(t) + \phi^{-1}(\frac{p}{q}\varepsilon^{\frac{p-q}{q}}b(t)) \int_0^t k(s)z(s)ds, \quad t \in [0, \infty), \tag{2.33}$$
where $\widetilde{a}$ and k are defined as in Theorem 2.4.3.

Applying lemma 2.3.2 for $p = 1$ to inequality (2.33) and using (2.29) we get the required inequality in (2.27). ■

2.4.2 New refinements of fractional integral inequalities

In this subsection, refinements of fractional integral inequalities are presented, in which the right-hand side contains a nonlinear fractional integral term involving class $\mathfrak{F}$ functions.

Theorem 2.4.4 *Let $\beta \in (0, 1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0, +\infty)$, such that $a(t)(a(t) \neq 0)$ is nondecreasing function, $l(t)$ be a nonnegative and continuous function on $(0, +\infty)$ with $t^{-\gamma}l(t)L(t, a^2(t))L^q_{Loc}[0, +\infty)(q > \frac{1}{\beta})$, and*

$u(t)$ be a continuous, nonnegative function on $[0,+\infty)$. Let $f:\mathbb{R}_+^2\to\mathbb{R}_+$ be a continuous function such that

$$0\leq f(t,x)-f(t,y)\leq L(t,y)(x-y),\quad x\geq y\geq 0, \tag{2.34}$$

for $t\in\mathbb{R}_+$ and $x\geq y\geq 0$, where $L:\mathbb{R}_+^2\to\mathbb{R}_+$ is a continuous function . If

$$u(t)\leq a(t)+b(t)\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)f(s,u(s))ds,\quad t\in[0,\infty), \tag{2.35}$$

then

$$u(t)\leq a(t)\left\{1+b(t)(A(t)+B(t)\int_0^t R(s)A(s)\exp(\int_s^t R(\tau)B(\tau)d\tau)ds)^{\frac{1}{q}}\right\},\ t\in[0,+\infty), \tag{2.36}$$

where

$A(t)=2^{q-1}\tilde{a}^q(t),\quad B(t)=\frac{2^{q-1}t^{q\beta-q+\frac{p}{q}}b^q(t)}{(p\beta--p+1)^{\frac{q}{p}}}t^{\beta-1+\frac{1}{q}},\quad R(t)=t^{-q\gamma}r^q(t),p\in(1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1,$

and

$\tilde{a}(t)=\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)\frac{1}{a(s)}f(s,a^2(s))ds$,$r(t)=l(t)L(t,a^2(t))$.

Proof. The inequality (2.35) can be written as :

$$\frac{u(t)}{a(t)}\leq 1+b(t)\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)\frac{f(s,a(s)\frac{u(s)}{a(s)})}{a(s)}ds. \tag{2.37}$$

Setting $w(t)=\frac{u(t)}{a(t)}$, one can reformulate (2.37) as

$$w(t)\leq 1+b(t)\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)\frac{f(s,a(s)w(s))}{a(s)}ds.$$

Let $g(t,w(t))=\frac{1}{a(t)}f(t,a(t)w(t))$, it is easy to see that $g(t,x)$ satisfies :

$$g(t,x)-g(t,y)\leq L(t,a(t)y)(x-y), \tag{2.38}$$

then

$$w(t)\leq 1+b(t)\int_0^t(t-s)^{\beta-1}s^{-\gamma}l(s)g(s,w(s))ds, \tag{2.39}$$

Applying Theorem 2.4.1 to the inequality (2.39),we get the required inequality in (2.36) ■

Theorem 2.4.5 *Let $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,T)$ $(0 < T \leq +\infty), l(t) \in L^q_{Loc}[0,T) \cap C[0,T[(q > 1)$. Let $g \in C[0,+\infty[$ be a nondecreasing, nonnegative function. and $u(t)$ be a continuous and nonnegative function on $[0,T)$. If*

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(u(s))\, ds, \quad t \in [0,\infty). \tag{2.40}$$

Then

$$u(t) \leq \left\{ \Omega_q^{-1} \left[\Omega_q(\widetilde{a}(t)) + \widetilde{b}(t) \int_0^t l^q(s) ds \right] \right\}^{\frac{1}{q}} \quad t \in [0, T_1]. \tag{2.41}$$

where

$$\begin{aligned} \widetilde{a}(t) &= 2^{q-1} a(t), \\ \widetilde{b}(t) &= 2^{q-1} (t^\theta \beta [-p\gamma+1,\ p(\beta-1)+1])^{\frac{q}{p}} b^q(t), \\ \Omega_q(\upsilon) &= \int_{\upsilon_0}^{\upsilon} \frac{ds}{g^q\left(s^{\frac{1}{q}}\right)}, \end{aligned} \tag{2.42}$$

and $T_1 \in (0,T)$ is such that $\Omega_q(\widetilde{a}(t)) + \widetilde{b}(t) \int_0^t l^q(s)) ds \in Dom(\Omega_q^{-1})$.

Proof. Let

$$z(t) = a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(u(s)) ds,$$

then

$$\begin{aligned} u(t) &\leq z(t), \\ z(t) &\leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(z(s)) ds. \end{aligned} \tag{2.43}$$

Let $1 \leq p < +\infty$, such tthat $\frac{1}{p} + \frac{1}{q} = 1$.Using the Höder inequality, we obtain from (2.43)

$$z(t) \leq a(t) + b(t) \left[\int_0^t (t-s)^{p(\beta-1)} s^{-p\gamma} ds \right]^{\frac{1}{p}} \left[\int_0^t l^q(s) g(z(s))^q\, ds \right]^{\frac{1}{q}}. \tag{2.44}$$

Since $(A+B)^n \leq 2^{n-1}(A^n + B^n)$ holds for any $A \geq 0, B \geq 0$ and using Lemma 1.4.7

$$\int_0^t (t-s)^{p(\beta-1)} s^{-p\gamma} ds = t^\theta \beta [-p\gamma+1,\ p(\beta-1)+1]\ , t \in \mathbb{R}_+, \tag{2.45}$$

we obtain from (2.44) that

$$z^q(t) \leq 2^{q-1} a(t) + 2^{q-1} (t^\theta \beta [-p\gamma+1,\ p(\beta-1)+1])^{\frac{q}{p}} b^q(t) \int_0^t l^n(s) g^q(z(s))\, ds,$$

where $\theta = p\left[(\beta-1)-\gamma\right]+1$.

Then the above inequality can be reformulated as

$$z^{q}(t) \leq \widetilde{a}(t)+\widetilde{b}(t)\int_{0}^{t} l^{q}(s)g^{q}(z(s))\,ds. \tag{2.46}$$

Let $t^{*}\in[0,t]$ be a positive constant chosen, we get

$$z^{q}(t) \leq \widetilde{a}(t^{*})+\widetilde{b}(t^{*})\int_{0}^{t} l^{q}(s)g^{q}(z(s))\,ds. \tag{2.47}$$

Let $G(t)$ be the right-hand side of the inequality (2.47) ,then $z(t)\leq G^{\frac{1}{q}}(t)$ and this yields $g^{q}(z(t))\leq g^{q}\left(G^{\frac{1}{q}}(t)\right)$. It is clear that

$$\frac{G'(t)}{g^{q}\left(G^{\frac{1}{q}}(t)\right)} = \frac{\widetilde{b}(t^{*})l^{q}(t)g^{q}(z(t))}{g^{q}\left(G^{\frac{1}{q}}(t)\right)},$$

i.e.,

$$\frac{d}{dt}\int_{0}^{G(t)}\frac{d\sigma}{g^{q}\left(\sigma^{\frac{1}{q}}\right)} \leq \widetilde{b}(t^{*})l^{q}(t),$$

or

$$\frac{d}{dt}\Omega_{q}(G(t)) \leq \widetilde{b}(t^{*})l^{q}(t). \tag{2.48}$$

where Ω_{q} is defined by (2.42) .

Integrating the inequality(2.48) from 0 to t, we obtain

$$\Omega_{q}\left(z(t)^{q}\right) \leq \Omega_{q}(\widetilde{a}(t^{*}))+\widetilde{b}(t^{*})\int_{0}^{t} l^{q}(s)ds, \tag{2.49}$$

Letting $t=t^{*}$ in (2.49) and considering $t^{*}>0$ is arbitary, after substituting t^{*} with t, we get

$$\Omega_{q}\left(z(t)^{q}\right) \leq \Omega_{q}(\widetilde{a}(t))+\widetilde{b}(t)\int_{0}^{t} l^{q}(s)ds. \tag{2.50}$$

Then

$$z(t) \leq \left\{\Omega_{q}^{-1}\left[\Omega_{q}(\widetilde{a}(t))+\widetilde{b}(t)\int_{0}^{t} l^{q}(s)ds\right]\right\}^{\frac{1}{q}}. \tag{2.51}$$

■

Inspired by the concept of inequality (2.40), one can derive a bound of an fractional integral inequality in the next corollary using functions of class $\mathfrak{F}$.

Corollary 2.4.4 *Let $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,T)$ $(0 < T \leq +\infty)$, $l(t) \in L^q_{Loc}[0,T)(q > 1)$.Let $g \in C[0,+\infty[$ belongs to class $\mathfrak{F}$ (see Definition 1.4.1), and $a(t) \neq 0$ be nondecreasing function in $[0,X)$ and $u(t)$ be a continuous and nonnegative function on $[0,T)$. If*

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) g(u(s))\, ds, \quad t \in [0,\infty). \tag{2.52}$$

Then

$$u(t) \leq a(t) \left\{ \Omega_q^{-1} \left[\Omega_q(2^{q-1}) + \widetilde{b}(t) \int_0^t l^q(s) ds \right] \right\}^{\frac{1}{q}} \quad t \in [0,T_1]. \tag{2.53}$$

where $\widetilde{b}(t), \Omega_q$ are defined as in Theorem 2.4.5.

Proof. The inequality (2.52) can be rewritten as

$$\frac{u(t)}{a(t)} \leq 1 + \frac{1}{a(t)} b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} g(u(s))\, ds \ . \tag{2.54}$$

Since $a(x)$ is nondecreasing function, we get

$$\frac{u(t)}{a(t)} \leq 1 + b(t) \int_0^t \frac{1}{a(s)} \left[(x-s)^{\beta-1} s^{-\gamma} l(s) g(u(s)) \right] ds. \tag{2.55}$$

Let $z(t) = \frac{u(t)}{a(t)}$ .Since g belongs to class $\mathfrak{F}$, one has

$$z(t) \leq 1 + b(t) \int_0^t \left[(x-s)^{\beta-1} s^{-\gamma} l(s) g(z(s)) \right] ds, \tag{2.56}$$

the rest of the proof is identical to the proof of the Theorem 2.3.5. ■

2.5 Applications

In this section, we present some examples aimed at studying and exploring certain properties of the solutions to the following initial value problem :

$$\begin{cases} D_r^\beta x(t) = f(t, x(t)) \quad t \in (0,\infty), \quad \beta \in (0,1) \\ \lim_{t \to 0^+} t^{1-\beta} x(t) = x_0, \end{cases} \tag{2.57}$$

where D_r^β is the Riemann–Liouville fractional derivative and the function f satisfies certain inequalities.

Theorem 2.5.1 ([23]) *Let* $f(t,x)$ *be a function that is continuous on the set*

$$B = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, x \in I\},$$

where $I \subseteq \mathbb{R}$ *denotes an unbounded interval. Suppose a function* $x : (0,T] \to I$ *is continuous and that both* $x(t)$ *and* $f(t,x(t))$ *are absolutely integrable on* $(0,T]$. *Then* $x(t)$ *satisfies the initial value problem* (2.57) *on* $(0,T]$ *if and only if it satisfies the Volterra integral equation*

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds. \tag{2.58}$$

Lemma 2.5.1 ([108]) *Suppose* $f : (0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, and there exist nonnegative functions* $l(t), k(t)$ *with* $t^{\beta-1}l(t) \in C(0,T] \cap L^q[0,T]$ *and* $k(t) \in C(0,T] \cap L^q[0,T]$ $(q > \frac{1}{\beta}, \beta \in (0,1))$ *such that*

$$| f(t,x) | \leq l(t) | x | + k(t),$$

for all $(t,x) \in (0,T] \times \mathbb{R}$.*Then the Volterra integral equation* (2.58) *has at least one continuous solution in* $C_{1-\beta}(0,T]$.

Proof. *Let* $G : C_{1-\beta}(0,T] \to C_{1-\beta}(0,T]$ *be the operator defined by*

$$Gx(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds \tag{2.59}$$

for all $x \in C_{1-\beta}(0,T]$.

Step 1 : *we show that the operator* G *is continuous. To see this let* $x_n \to x$ *in* $C_{1-\beta}(0,T]$ *and we will show that* $Gx_n \to Gx$ *in* $C_{1-\beta}(0,T]$. *Now* $x_n \to x$ *implies that there exists* $r > 0$ *such that* $\| x_n \|_{1-\beta} \leq r$ *and* $\| x \|_{1-\beta} \leq r$. *For each* $s \in (0,T]$, *we have*

$$f(s,x_n(s)) \to f(s,x(s)).$$

Using the assumption of f, *we ge*

$$(t-s)^{\beta-1} | f(s,x_n(s)) - f(s,x(s)) | \leq 2(t-s)^{\beta-1}(rs^{\beta-1}l(s) + k(s)).$$

Since $t^{\beta-1}l(t) \in C(0,T] \cap L^q[0,T]$ *and* $k(t) \in C(0,T] \cap L^q[0,T]$, *using the Hölder inequality, then we know the function*

$$s \to 2r(t-s)^{\beta-1}s^{\beta-1}l(s) + 2(t-s)^{\beta-1}k(s)$$

is integrable for $s \in [0,t]$. *By means of the Lebesgue dominated convergence theorem yields*

$$t^{1-\beta} \mid \int_0^t (t-s)^{\beta-1}[f(s,x_n(s)) - f(s,x(s))] \mid ds \to 0$$

as $n \to +\infty$. *Therefore* $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ *pointwise on* $[0,T]$ *as* $n \to +\infty$. *If we show the convergence is uniform then of course* G *is continuous. Let* $t_1, t_2 \in [0,T]$ *with* $t_2 < t_1$.

Then

$$\begin{aligned} &\mid t_1^{1-\beta}Gx_n(t_1) - t_2^{1-\beta}Gx(t_2) \mid \\ &\leq \mid \frac{t_1^{1-\beta} - t_2^{1-\beta}}{\Gamma(\beta)} \mid\mid \int_0^{t_2}(t_2-s)^{\beta-1}f(s,x(s))ds \mid \\ &+ \frac{t_1^{1-\beta}}{\Gamma(\beta)} \mid \int_0^{t_1}(t_1-s)^{\beta-1}f(s,x(s))ds - \int_0^{t_2}(t_2-s)^{\beta-1}f(s,x(s))ds \mid . \end{aligned} \tag{2.60}$$

Since

$$\mid f(t,x(t)) \mid \leq l(t) \mid x(t) \mid + k(t) \leq t^{\beta-1}l(t)t^{1-\beta} \mid x(t) \mid + k(t),$$

from the assumptions of f, *we know* $f(t,x(t)) \in L^q[0,T](q > \frac{1}{\beta})$ *when* $x(t) \in C_{1-\beta}(0,T]$.*From* ***Lemma 1.4.5***, *we obtain*

$$\int_0^t (t-s)^{\beta-1}f(s,x(s))ds$$

is continuous on $[0,T]$. *As* $t_1 \to t_2$, *the right-hand side of the above inequality* (2.60) *tends to zero. Now* (2.60) *together with the fact that* $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ *pointwise on* $[0,T]$ *implies that the convergence is uniform. Consequently* $G : C_{1-\beta}(0,T] \to C_{1-\beta}(0,T]$ *is continuous.*

Step 2 : *Next we claim that the operator* G *is completely continuous. To see this let* $\Omega \in C_{1-\beta}(0,T]$ *be bounded and* $\| x_n \|_{1-\beta} \leq M$ *for each* $x \in \Omega$, *we will show that* $t^{1-\beta}G(\Omega)$

is uniformly bounded and equicontinuous on $[0,T]$. *The equicontinuity of* $t^{1-\beta}G(\Omega)$ *on* $[0,T]$ *follows essentially the same reasoning as that used to prove* (2.60). *Also* $t^{1-\beta}G(\Omega)$ *is uniformly bounded. Since for* $t\in[0,T]$, *we have*

$$\begin{aligned} \mid t^{1-\beta}Gx(t)\mid \ &\leq\mid x_0\mid + \tfrac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}\upsilon(s)s^{1-\beta}\mid x(s)\mid ds+\tfrac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}k(s)ds\\ &\leq\mid x_0\mid + \tfrac{t^{1-\beta}}{\Gamma(\beta)}(\int_0^t(t-s)^{p(\beta-1)}ds)^{\frac{1}{p}}(\int_0^t Ms^{\beta-1}\upsilon(s))^q ds)^{\frac{1}{q}}\\ &\quad+\tfrac{t^{1-\beta}}{\Gamma(\beta)}(\int_0^t(t-s)^{p(\beta-1)ds})^{\frac{1}{p}}(\int_0^t k^q(s)ds)^{\frac{1}{q}}\\ &\leq\mid x_0\mid + \tfrac{t^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{P}}}[(\int_0^t Ms^{\beta-1}\upsilon(s))^q ds)^{\frac{1}{q}}+(\int_0^t k^q(s)ds)^{\frac{1}{q}}], \end{aligned} \tag{2.61}$$

then

$$\parallel Gx\parallel_{1-\beta}\leq\mid x_0\mid+\frac{T^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{P}}}[(\int_0^T Ms^{\beta-1}\upsilon(s))^q ds)^{\frac{1}{q}}+(\int_0^T k^q(s)ds)^{\frac{1}{q}}].$$

Consequently $G: C_{1-\beta}(0,T]\to C_{1-\beta}(0,T]$ *is completely continuous.*

Step $3:$ ***If*** $x\in C_{1-\beta}(0,T]$ *is any solution of*

$$x(t)=\lambda(x_0t^{\beta-1}+\frac{1}{\Gamma(\beta)}(\int_0^t(t-s)^{\beta-1}f(s,x(s))ds,\ t\in(0,T]$$

for $\lambda\in(0,1)$. *Let* $\upsilon(t)=t^{1-\beta}x(t)\in C[0,T]$, *then*

$$\begin{aligned} \mid\upsilon(t)\mid\ &\leq\mid x_0\mid+\mid\tfrac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f(s,s^{\beta-1}\upsilon(s))ds\mid\\ &\leq\mid x_0\mid+\tfrac{T^{\frac{1}{P}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{P}}}(\int_0^t k^q(s)ds)^{\frac{1}{q}}+\tfrac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}l(s)\mid\upsilon(s)\mid ds. \end{aligned} \tag{2.62}$$

Consequently, by ***Theorem 2.3.6*** *we can get*

$$\mid\upsilon(t)\mid\leq(Q(t)+W(t)\int_0^t M(s)Q(s)\exp(\int_s^t M(\tau)W(\tau)d\tau)ds)^{\frac{1}{q}},\quad t\in[0,T],$$

where

$$Q(t) = 2^{q-1}(\mid x_0 \mid + \frac{T^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{P}}}(\int_0^t k^q(s)ds)^{\frac{1}{q}})^q,$$
$$W(t) = \frac{2^{q-1}t^{\frac{q}{p}}}{\Gamma^q(\beta)(p\beta-p+1)^{\frac{q}{P}}},$$
$$M(t) = t^{q(\beta-1)}\upsilon^q(t)$$

and $p \in (1, +\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then we get*

$$\parallel \upsilon \parallel \leq \parallel x \parallel_{1-\beta} \leq (Q(T) + W(T)\int_0^T M(s)Q(s)\exp(\int_s^T M(\tau)W(\tau)d\tau)ds)^{\frac{1}{q}}.$$

Finally, by applying fixed point ***Theorem 1.4.1***, *the operator* G *has a fixed point* $x(t)$ $\in C_{1-\beta}(0, T]$ *which is the solution of the integral equation* (2.58). ■

Theorem 2.5.2 *If* $f : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, and*

$$|f(t, x) - f(t, y)| \leq l(t)g\left(|x - y|\right), \tag{2.63}$$

for all $x, y \in \mathbb{R}$ *and* $t \in (0, +\infty)$,*where* g *is defined as in corollary* 2.4.1 *such that* $g(0) = 0$, $l(t) \in C(0, +\infty) \cap L^q_{Loc}[0, +\infty)$ *and* $|f(t, 0)| \in L^q_{Loc}[0, +\infty)$ $(q > \frac{1}{\beta})$.*Then equation* (2.58) *has a unique global solution on* $(0, +\infty)$

Proof. *We know*

$$|f(t, x)| \leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq l(t)g(|x|) + |f(t, 0)|.$$

Applying the mean value Theorem for the function g, *then for every* $|x| > 0$, *there exists* $c \in]\ 0, |x|[$*such that*

$$g(|x|) - g(0) = g^{'}(c)(|x| - 0) \leq g^{'}(0)(|x| - 0),$$

then

$$|f(t, x)| \leq l(t)g^{'}(0)(|x|) + |f(t, 0)|.$$

By Lemma 2.4.1, *the equation* (2.58) *has at least one global solution* .

Now, suppose $x_1(t)$, $x_2(t)$ *are two global solutions of equation* (2.58), *then we have*

$$
\begin{aligned}
x_1(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(f(s,x_1(s))ds \\
x_2(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(f(s,x_2(s)))ds, t\in[0,\infty)
\end{aligned}
$$

Then , we have

$$x_1(t)-x_2(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(f(s,x_1(s))-f(s,x_2(s)))ds$$

which implies that

$$\mid x_1(t)-x_2(t)\mid\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\beta-1}s^{1-\beta}l(s)g(\mid x_1(s)-x_2(s)\mid)ds. \tag{2.64}$$

Let $u(t)=\mid x_1(s)-x_2(s)\mid, L(t)=s^{1-\beta}l(s)$*,then*

$$u(t)\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\beta-1}L(s)g(u(s))ds. \tag{2.65}$$

By corollary 2.4.1, we can get $x_1(t)=x_2(t)$*. Thus the proof is complete.* ■

Example 2.5.1

$$\begin{cases} D_r^{\frac{4}{5}}x(t)=t^{\frac{-7}{12}}\arctan(x)+t^{\frac{-1}{2}}, \\ \lim_{t\to 0^+} t^{\frac{1}{5}}x(t)=1. \end{cases} \tag{2.66}$$

We know that

$$\mid f(t,x)-f(t,y)\mid=\left|t^{\frac{-7}{12}}\arctan(x)-t^{\frac{-7}{12}}\arctan(y)\right|=t^{\frac{-7}{12}}\left|\arctan(\frac{x-y}{1+xy})\right|\leq t^{\frac{-7}{12}}\arctan(\mid x-y\mid),$$

where $x,y\in(0,+\infty)$*. Since* $t^{\frac{-7}{12}}\in C(0,+\infty)\cap L^q_{Loc}[0,+\infty)$ *and* $t^{\frac{-1}{2}}\in C(0,+\infty)\cap L^q_{Loc}[0,+\infty)$ $(q>\frac{5}{4})$*, then from Theorem 2.4.7, equation (2.66) has a unique global solution on* $(0,+\infty)$*.*

CHAPTRE 3

On Some Pachpatte–Gamidov-Type Integral Inequalities and Their Discrete Analogues

3.1 Introduction

This chapter is entirely devoted to Gamidov-type inequalities. These inequalities were first introduced by Gamidov [53] in 1969 in the context of boundary value problems involving higher-order differential equations. Since then, many researchers such as Bainov, Pachpatte, Kendre, and Latpate have developed more general and refined versions of these inequalities, [53] .These contributions are now regarded as fundamental tools for analyzing the qualitative behavior of solutions to certain classes of integral equations. For further details, we can refer to [9, 39, 40, 65, 89, 91, 93].

The structure of this chapter is as follows:

In the first section, we introduce the classical Gamidov inequalities.

In the second section, we present some generalizations found in the literature that have proven useful in various areas of differential equations.

In the final section, we establish new Gamidov-type inequalities on arbitrary time scales. These results are part of an article currently under consideration for publication in an international journal.

3.2 Well-known Gamidov-type integral inequalities

In this first part, we present the Gamidov-type inequalities.

In 1969 Gamidov [53] established the following inequalities to apply them in the study of certain boundary value problems.

In all that follows, we will designate the interval $[\alpha, \beta]$ by J .

Corollary 3.2.1 ([53]) *Let $u,\ f,\ g_1,\ g_2,\ h_i\ (i = 1, 2, ...n)$ be positive, continuous functions defined on J. If*

$$u(t) \leq f(t) + g_1(t)\int_{t_1}^{t} h_1(s)u(s)ds + g_2(t)\sum_{i=1}^{n} c_i \int_{t_1}^{t_i} h_i(s)u(s)ds,$$

where

$$\alpha = t_1 \leq t_2 \leq .... \leq t_n = \beta \text{ and } c_i \text{ are constants,}$$

with

$$\sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)\left[g_2(s) + g_1(s)\int_{t_1}^{s} h_1(\tau)g_2(\tau) \times \exp\left(\int_{\tau}^{s} g_1(\tau)\, h_1(\sigma)\, d\sigma\right) d\tau\right] ds < 1,$$

then , we have

$$u(t) \leq p_1(t) + Mp_2(t),$$

where

$$p_1(t) = f(t) + g(t)\int_{t_1}^{t} h_1(s)f(s)\exp\left(\int_{s}^{t} g_1(\sigma)h_1(\sigma)d\sigma\right) ds,$$
$$p_2(t) = g_2(t) + g_1(t)\int_{t_1}^{t} h_1(s)g_2(s)\exp\left(\int_{s}^{t} g_1(\sigma)\, h_1(\sigma)d\sigma\right) ds,$$

$$M = \left(\sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)p_1(s)ds\right)\left(1 - \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)p_2(s)ds\right)^{-1}.$$

Theorem 3.2.1 ([53]) ([37]) *Let u and k be continuous positive functions on J, and let $0 < p < 1$, $a_1 \geq 0$, $a_2 \geq 0$, and $a_3 > 0$ be constants. Suppose that*

$$u(t) \leq a_1 + a_2 \int_{\alpha}^{t} k(s)u^p(s)ds + a_3 \int_{\alpha}^{\beta} k(s)u^p(s)ds,\ t \in J,$$

then, we have

$$u(t) \leq \left(x_0^q + a_2 q \int_{\alpha}^{t} k(s)ds \right)^{\frac{1}{q}},\ t \in J,$$

where $q = 1 - p$ *and* x_0 *is the unique positive root of the equation*

$$\left[\frac{a_2 + a_3}{a_3} x - \frac{a_1 a_2}{a_3} \right]^q - x^q - a_2 q \int_{\alpha}^{\beta} k(s)ds = 0.$$

In 1989, **Leela and Martynyuk** [70] established a new Gamidov-type inequality, which is stated as follows:

Theorem 3.2.2 ([70]) *Let $m, v \in C(\mathbb{R}_+, \mathbb{R}_+)$, $T > t_0 \geq 0$. If the inequality*

$$m(t) \leq m(t_0) + \int_{t_0}^{t} v(s)m(s)ds + \int_{t_0}^{T} v(s)m(s)ds, \quad \forall\ t_0 \leq t \leq T,$$

is satisfied, and

$$\exp\left(\int_{t_0}^{T} v(s)ds \right) < 2,$$

then ,we get

$$m(t) \leq \frac{m(t_0)}{2 - \exp\left(\int_{t_0}^{T} v(s)ds \right)} \exp\left(\int_{t_0}^{t} v(s)ds \right).$$

In 1992, **Bainov and Simeonov [9]** Bainov established a variant of Theorem 3.2.2. The result is stated below:

Theorem 3.2.3 ([9]) *Let u, b, and c be continuous functions on J such that b et c are positive on J. Let us admit that*

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)ds + \int_{\alpha}^{\beta} c(s)u(s)ds,\ t \in J,$$

where a is a constant and if

$$q = \int_{\alpha}^{\beta} C(s) \exp\left(\int_{\alpha}^{s} b(\tau)\, d\tau \right) ds < 1,$$

then,

$$u(t) \leq \frac{a}{1-q} \exp\left(\int_{\alpha}^{t} b(s)ds \right),\ t \in J.$$

In [91] , Pachpatte established a more general version of Gamidov's inequality. In the statement below, we denote by $\Delta = \{(t,s) \in J^2 : \alpha \leq s \leq t \leq \beta\}$.

Theorem 3.2.4 ([*91*]) *Let* $u \in C(J, \mathbb{R}_+)$, $a, b \in C(\Delta, \mathbb{R}_+)$ *such that* a *and* b *are increasing functions in* t, *for each* $s \in J$ *and suppose that*

$$u(t) \leq c + \int_\alpha^t a(t,s)u(s)ds + \int_\alpha^\beta b(t,s)u(s)ds,$$

for all $t \in J$, *where* $c \geq 0$ *is a constant. If the condition*

$$p(t) = \int_\alpha^\beta b(t,s) \exp\left(\int_\alpha^s a(s,\sigma)d\sigma\right) ds < 1,$$

is satisfied, then we have

$$u(t) \leq \frac{c}{1-p(t)} \exp\left(\int_\alpha^t a(t,s)ds\right), \ \forall t \in J.$$

3.3 various generalizations

We will present, without demonstrations, some variants and generalizations obtained on the inequalities presented in the previous section. For more details we can consult [91, 92, 65, 9, 40, 32].

Theorem 3.3.1 ([92]) (H_1) *Let* a *be a continuously differentiable function on* J *such that* $a'(t) \geq 0$. *Assume that*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s)ds + \int_\alpha^\beta c(s)u(s)ds, \ \forall t \in J,$$

and if

$$p_1 = \int_\alpha^\beta c(s) \exp\left(\int_\alpha^s b(\sigma)d\sigma\right) ds < 1,$$

then

$$u(t) \leq M_1 \exp\left(\int_\alpha^t b(s)ds\right) + \int_\alpha^t a'(s) \exp\left(\int_s^t b(\sigma)d\sigma\right) ds,$$

where

$$M_1 = \frac{1}{1-p_1}\left[a(\alpha) + \int_\alpha^\beta c(s)(\int_\alpha^s a'(\tau) \exp\left(\int_\tau^s b(\sigma)d\sigma\right) d\tau)ds\right].$$

(H_2) Suppose that

$$u(t) \leq a(t) + b(t) \int_{\alpha}^{t} f(s)u(s)ds + c(t) \int_{\alpha}^{\beta} g(s)u(s)ds, \ \forall t \in J,$$

and if we have

$$p_2 = \int_{\alpha}^{\beta} g(s)k_2(s)ds < 1,$$

then

$$u(t) \leq k_1(t) + M_2 k_2(t), \forall t \in J,$$

with

$$k_1(t) = a(t) + b(t) \int_{\alpha}^{t} f(\tau)a(\tau) \exp\left(\int_{\tau}^{t} f(\sigma) b(\sigma)d\sigma \right) d\tau,$$
$$k_2(t) = c(t) + b(t) \int_{\alpha}^{t} f(\tau)c(\tau) \exp\left(\int_{\tau}^{t} f(\sigma) b(\sigma)d\sigma \right) d\tau,$$

and

$$M_2 = \frac{1}{1-p_2} \int_{\alpha}^{\beta} g(s)k_1(s)ds.$$

(H_3) Let $h(t,s)$ and its partial derivative $\frac{\partial h(t,s)}{\partial t}$ be positive and continuous functions for $\alpha \leq s \leq t \leq \beta$ and if the inequality

$$u(t) \leq a(t) + \int_{\alpha}^{t} h(t,s)u(s)ds + \int_{\alpha}^{\beta} c(s)u(s)ds,$$

is satisfied for all $t \in J$, moreover if

$$P_3 = \int_{\alpha}^{\beta} c(s) \exp\left(\int_{\alpha}^{s} B^*(\sigma) \, d\sigma \right) ds < 1,$$

then,

$$u(t) \leq a(t) + M_3 \exp\left(\int_{\alpha}^{t} B^*(\sigma) \, d\sigma \right) + \int_{\alpha}^{t} A^*(s) \exp\left(\int_{s}^{t} B^*(\sigma) \, d\sigma \right) ds \ , \ \forall t \in J,$$

where

$$A^*(t) = h(t,t)a(t) + \int_{\alpha}^{t} \frac{\partial}{\partial t} h(t,s)a(s)ds \ ,$$
$$B^*(t) = h(t,t) + \int_{\alpha}^{t} \frac{\partial}{\partial t} h(t,s)ds,$$

and

$$M_3 = \frac{1}{1-P_3} \int_{\alpha}^{\beta} c(s) \left[a(s) + \int_{\alpha}^{s} A^*(\tau) \exp\left(\int_{\tau}^{s} B^*(\sigma) \, d\sigma \right) d\tau \right] ds.$$

Remark 3.3.1 *The inequality established in (H_1, Theorem 3.3.1) is a variant of the inequality given by Bainov and Simeonov in (Theorem 3.2.3), while the inequality established in (H_2) is a variant of the inequality given by Gamidov in Corollary 3.2.1.*

Kendre and Latpate [65] extended the inequality proved by Pachpatte in (Theorem 3.3.1, (H_1)) by establishing the following results:

Theorem 3.3.2 ([65]) *Let $u, f, g, c, c' \in C(J, \mathbb{R}_+)$ and $p \geq q \geq 0$, $p \neq 0$ are constants.*

If the inequality

$$u^p(t) \leq c(t) + \int_\alpha^t f(s)u^q(s)ds + \int_\alpha^\beta g(s)u^p(s)ds,$$

is satisfied, and

$$\varphi = \int_\alpha^\beta g(s) \exp\left(\int_\alpha^s n_1 f(\sigma)d\sigma\right) ds < 1,$$

then, we have

$$\begin{aligned} u^p(t) \leq & \frac{c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2 f(\tau)] \exp\left(\int_\tau^s n_1 f(\sigma)d\sigma\right) d\tau\right) ds}{1-\varphi} \\ & \times \exp\left(\int_\alpha^t m f(s)ds\right) + \int_\alpha^t [c'(s) + n_2 f(s)] \exp\left(\int_s^t n_1 f(\sigma)\, d\sigma\right) ds, \end{aligned}$$

where

$$k > 0,\ n_1 = \tfrac{q}{p} k^{\frac{q-p}{p}} \qquad et \qquad n_2 = \tfrac{p-q}{p} k^{\frac{q}{p}}.$$

Theorem 3.3.3 ([65]) *Let $u, f, g, h \in C(J, \mathbb{R}_+)$ and $c \geq 0$ is a constant. If the inequality*

$$u^p(t) \leq c + \int_\alpha^t h(s)\left[u^q(s) + \int_\alpha^s f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)u^p(\sigma)d\sigma\right] ds, \ \forall t \in J,$$

then

$$u^p(t) \leq c \exp\left(\int_\alpha^t n_1 A(\sigma)d\sigma\right) + \int_\alpha^t n_2 B(s) \exp\left(\int_s^t n_1 A(\sigma)\, d\sigma\right),$$

where

$$\begin{aligned} A(t) &= h(t)\left[1 + \int_\alpha^t f(\sigma)d\sigma + \frac{1}{n_1}\int_\alpha^\beta g(\sigma)d\sigma\right], \\ B(t) &= h(t)\left[1 + \int_\alpha^t f(\sigma)d\sigma\right], \end{aligned}$$

and p, q, n_1, n_2 are exactly the same defined in Theorem 3.3.2.

Corollary 3.3.1 ([65]) *Suppose that the assumptions of Theorem* 3.2.3 *are verified and let* $c(t) \geq 1$ *be an increasing function. If the inequality*

$$u^p(t) \leq c^p(t) + \int_\alpha^t h(s) \left[u^q(s) + \int_\alpha^s f(\sigma) u^q(s) d\sigma + \int_\alpha^\beta g(\sigma) u^p(\sigma) d\sigma \right] ds, \ \forall t \in J,$$

is satisfied, then

$$u^p(t) \leq c^p(t) \exp \int_\alpha^t n_1 A(\sigma) d\sigma + c^p(t) \int_\alpha^t n_2 B(s) \exp \left(\int_s^t n_1 A(\sigma) d\sigma \right),$$

where $p \geq q \geq 1$, A, B *and* n_1, n_2 *are exactly the same functions defined in Theorem* 3.3.3 *and Theorem* 3.3.2 *respectively.*

Boukerrioua and Meziri obtained more refined results and more general versions of the inequalities presented in the previous two sections, extended to arbitrary time scales.

Theorem 3.3.4 ([32]) *Let us assume that* u, c, $g \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}_+\right)$ *and* $c^\Delta \geq 0$. *If* f *is defined as in Theorem 1.1.7 such that* $f(t,s) \geq 0$ *and* $f^\Delta(t,s) \geq 0$ *for* $t, s \in [a,b]_{\mathbb{T}}$ *with* $s \leq t$. *Then*

$$u^p(t) \leq c(t) + \int_a^t f(t,s) u^q(s) \Delta s + \int_a^b g(s) u^r(s) \Delta s,$$

implied

$$u(t) \leq \left[m(t) + \frac{\frac{r}{p} k^{\frac{r-p}{p}} e_P(t,a) \int_a^b g(s) m(s) \Delta s}{1 - \frac{r}{p} k^{\frac{r-p}{p}} \int_a^b g(s) e_P(s,a) \Delta s} \right]^{\frac{1}{p}},$$

for $t \in [a,b]_{\mathbb{T}}^k$, *where*

$$P(t) = \frac{q}{p} k^{\frac{q-p}{p}} \left[f(\sigma(t), t) + \int_a^t f^\Delta(t,s) \Delta s \right],$$

$$Q(t) = c^\Delta(t) + \frac{p-q}{p} k^{\frac{q}{p}} \left[f(\sigma(t), t) + \int_a^t f^\Delta(t,s) \Delta s \right],$$

and

$$m(t) = c(a) e_P(t,a) + \int_a^t Q(s) e_P(t, \sigma(s)) \Delta s + \frac{p-r}{p} k^{\frac{r}{p}} e_P(t,a) \int_a^b g(s) \Delta s,$$

provided that

$$\frac{r}{p}k^{\frac{r-p}{p}}\int_a^b g(s)e_P(s,a)\,\Delta s<1.$$

Remark 3.3.2 *If we take $\mathbb{T}=\mathbb{R}$, $p=q=r=1$, the inequality given in Theorem* 3.3.4 *reduces to the inequality given in [Theorem 3.3.1,(H_3)].*

Theorem 3.3.5 ([32]) *Let us assume that $u,\ h,\ g\in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_+)$, and $c\geq 0$ a constant. If f is defined as in Theorem 1.1.7 such that $f(t,s)\geq 0$ and $f^{\Delta}(t,s)\geq 0$ for $t,s\in[a,b]_{\mathbb{T}}$ with $s\leq t$. Then*

$$u^p(t)\leq c+\int_a^t h(s)\left[u^q(s)+\int_a^s f(s,\tau)u^q(\tau)\Delta\tau+\int_a^b g(\tau)u^r(\tau)\Delta\tau\right]\Delta s,$$

implied

$$u(t)\leq\left[c+\int_a^t h(s)\left(m(s)+\frac{l(s)\int_a^b g(\tau)m(\tau)\Delta\tau}{1-\int_a^b g(\tau)l(\tau)\Delta\tau}\right)\Delta s\right]^{\frac{1}{p}},$$

for $t\in[a,b]_{\mathbb{T}}^k$, provided that

$$\int_a^b g(\tau)l(\tau)\Delta\tau<1,$$

where

$$m(t)=(\frac{q}{p}K^{\frac{q-p}{p}}c+\frac{p-q}{p}K^{\frac{q}{p}}+\frac{p-r}{p}K^{\frac{r}{p}}\int_a^b g(\tau)\Delta\tau)e_P(t,a)+\int_a^t Q(\tau)e_P(t,\sigma(\tau))\Delta\tau,$$

and

$$\begin{aligned}P(t)&=\frac{q}{p}K^{\frac{q-p}{p}}h(t)+f(\sigma(t),t)+\int_a^t f^{\Delta}(t,\tau)\Delta\tau,\\ Q(t)&=\frac{p-q}{p}K^{\frac{q}{p}}(f(\sigma(t),t)+\int_a^t f^{\Delta}(t,\tau)\Delta\tau).\end{aligned}$$

Remark 3.3.3 *Note that when, $\mathbb{T}=\mathbb{R}$, $r=p$, $f(t,s)=f(t)$, Theorem* 3.3.5 *reduces to the inequality indicated in Theorem* 3.3.3.

3.4 Some Refinements of the Pachpette-Gamidov type inequalities on time scales

In this part, we will present and prove some new versions of Pachpette-Gamidov-type integral inequalities on time scales, which generalize the results obtained by [82] .

Before stating our new refinements, we present the following lemma due to its usefulness in our results.

Lemma 3.4.1 *Suppose* $u, m, l, n \in C([t_0, T], \mathbb{R}_+)$. *Let* $M : [t_0, T]_{\mathbb{T}}^k \times \mathbb{R}_+ \to \mathbb{R}_+$ *be an rd-continuous function satisfying*

$$0 \leq M(t, x) - M(t, y) \leq R(t, y)(x - y), \tag{3.1}$$

for $t \in [t_0, T]_{\mathbb{T}}^k$ and $x \geq y \geq 0$, where $R : [t_0, T]_{\mathbb{T}}^k \times \mathbb{R}_+ \to \mathbb{R}_+^*$ is an rd-continuous function. Then

If

$$u(t) \leq m(t) + l(t) \int_a^b n(s) M(s, u(s)) \Delta s,$$

Then

$$u(t) \leq m(t) + \frac{l(t) \int_a^b n(s) M(s, m(s)) \Delta s}{1 - \int_a^b R(s, m(s)) n(s) l(s) \Delta s}. \tag{3.2}$$

provided

$$\int_a^b R(s, m(s)) n(s) l(s) \Delta s < 1.$$

Proof. Let

$$\Pi = \int_a^b n(s) M(s, u(s)) \Delta s.$$

It's clear that Π is a constant, substituting the last inequality in (3.1), we get

$$u(t) \leq m(t) + l(t) \Pi.$$

From (3.1), we have

$$M\left(t,u(t)\right)\leq M(t,m(t)+l(t)\Pi)\leq M(t,m(t))+R(t,m(t))l(t)\Pi$$

Multiplying both sides of the above inequality by $n(t)$, then integrating the result from a to b, it yields

$$\int_{0}^{T} n(s)M\left(s,u(s)\right)\Delta s\leq \int_{a}^{b} n(s)M(s,m(s)\Delta s+\Pi\int_{0}^{T} R(s,m(s))n(s)l(s)\Delta s,$$

The above inequality can be rewrite as

$$\Pi\leq \int_{0}^{T} n(s)S(s,m(s)\Delta s+\Pi\int_{0}^{T} R(s,m(s))n(s)l(s)\Delta s,$$

the inequalities (3.9) implies the estimate

$$\Pi(1-\int_{0}^{T} R(s,m(s))n(s)l(s)\Delta s)\leq \int_{0}^{T} n(s)M(s,m(s))\Delta s,$$

then, we obtain

$$\Pi\leq \frac{\int_{0}^{T} n(s)S(s,m(s))\Delta s}{1-\int_{0}^{T} R(s,m(s))n(s)l(s)\Delta s}.$$

Therefore, from the last inequality, one can deduce inequality (3.2). ■

The main results are established in the following theorems.

Theorem 3.4.1 *Let $u(t),\ f(t),h(t),g(t),w(t)\ \in \mathcal{C}_{rd}\ ([a;b]_{\mathbb{T}},\ \mathbb{R}^{+})$, such that $h(t)$ is positive and nondecreasing function for all $t\in[a;b]_{\mathbb{T}}$. If the following inequality*

$$u(t)\leq f(t)+h(t)\int_{a}^{t} g(s)u(s)\Delta s+\int_{a}^{b} w(s)M(s,u(s))\Delta s \tag{3.3}$$

holds then $u(t)$ has the following estimate

$$u(t)\leq f(t)+h(t)(m(t)+\frac{l(t)\int_{a}^{b} w(s)M(s,m(s))\Delta s}{1-\int_{a}^{b} R(s,m(s))w(s)l(s)\Delta s}), \tag{3.4}$$

where

$$m(t) = h(t)\int_a^t g(\tau)f(\tau)e_{hg}(t,\sigma(\tau))\Delta\tau + f(t), \tag{3.5}$$
$$l(t) = \frac{h(t)e_{hg}(t,a)}{h(a)},$$

provided

$$\int_a^b R(s,m(s))w(s)l(s)\Delta s < 1. \tag{3.6}$$

Proof. Since $h(t)$ is positive and monotonic nondecreasing, we can restate (3.3) as follows :

$$u(t) \leq f(t) + h(t)\left[\int_a^t g(s)u(s)\Delta s + \frac{1}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s\right]. \tag{3.7}$$

Define a function $z(t)$ by

$$z(t) = \int_a^t g(s)u(s)\Delta s + \frac{1}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s. \tag{3.8}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq f(t) + h(t)z(t). \tag{3.9}$$

and

$$z(a) = \frac{1}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s \tag{3.10}$$

Differentiating (3.8) with respect to t, we obtain

$$z^{\Delta}(t) = g(t)u(t). \tag{3.11}$$

Substituting (3.9) into (3.11), we get

$$z^{\Delta}(t) \leq g(t)f(t) + h(t)g(t)z(t). \tag{3.12}$$

According **Lemma 1.4.2** to (3.12), we obtain

$$z(t) \leq z(a)e_{hg}(t,a) + \int_a^t g(\tau)f(\tau)e_{hg}(t,\sigma(\tau))\Delta\tau.$$

Using (3.10), and (3.12) in the above inequality, we get

$$z(t) \leq \int_a^t g(\tau)f(\tau)e_{hg}(t,\sigma(\tau))\Delta\tau + \frac{e_{hg}(t,a)}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s, \tag{3.13}$$

Multiplying both side of (3.13) by $h(t)$ and adding the result term $f(t)$, the above inequality can be restated as

$$u(t) \leq m(t) + l(t)\int_a^b w(s)M(s,u(s))\Delta s, \tag{3.14}$$

where $m(t)$ and $l(t)$ are defined as in (3.5),

then from **Lemma 3.4.1**, we have

$$u(t) \leq m(t) + \frac{l(t)\int\limits_a^b w(s)M(s,m(s))\Delta s}{1 - \int\limits_a^b R(s,m(s))w(s)l(s)\Delta s}. \tag{3.15}$$

■

Corollary 3.4.1 *Assume that all the assumptions of* ***Theorem 3.4.1*** *are satisfied . Let $G : \mathbb{R}_{+}\rightarrow\mathbb{R}_{+}$ is a differentiable increasing function on $\mathbb{R}_0$ with continuous non-increasing first derivative G' on $\mathbb{R}_0$.If*

$$u(t) \leq f(t) + h(t)\int_a^t g(s)u(s)\Delta s + \int_a^b w(s)G(u(s))\Delta s$$

then

$$u(t) \leq m(t) + \frac{l(t)\int\limits_a^b w(s)G(u(s))\Delta s}{1 - \int\limits_a^b w(s)l(s)G'(m(s))\Delta s}$$

for all $t \in I$, provided that

$$\int\limits_0^T w(s)l(s)G^{'}(m(s))ds < 1.$$

where

$$
\begin{aligned}
m(t) &= h(t)\int_a^t g(\tau)f(\tau)e_{hg}(t,\sigma(\tau))\Delta\tau + f(t), \\
l(t) &= \frac{h(t)e_{hg}(t,a)}{h(a)},
\end{aligned}
$$

Proof. Applying the mean value theorem for the function g then for every $x > y > 0$ there exists $c \in]\, y, x[$such that

$$G(x) - G(y) = G^{'}(c)(x-y) \leq G^{'}(y)(x-y),$$

then the function G satisfies (3.1), we omit the proof of **Lemma** 3.4.1.

Corollary 3.4.2 *Assume that all the assumptions of Theorem* 3.4.1 *are satisfied, and let* $\mathbb{T} = \mathbb{Z}$ *. If the following inequality*

$$u(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(\tau)u(\tau) + \sum_{s=a}^{b-1} w(s)M(s,u(s)),$$

holds, then $u(t)$ *has the following estimate*

$$u(t) \leq f(t) + h(t)(m(t) + \frac{l(t)\sum_{s=a}^{b-1} w(s)M(s,m(s))}{1 - \sum_{s=a}^{b-1} R(s,m(s))w(s)l(s)}),$$

where

$$
\begin{aligned}
m(t) &= h(t)\sum_{s=a}^{b-1} g(\tau)f(\tau)(\prod_{\tau=s+1}^{\tau=1} 1 + h(\tau)g(\tau)) + f(t), \\
l(t) &= \frac{h(t)(\prod_{\tau=s+1}^{\tau=1} 1 + h(\tau)g(\tau))}{h(a)},
\end{aligned}
$$

provided

$$\sum_{s=a}^{b-1} R(s,m(s))w(s)l(s)\Delta s < 1.$$

■

Theorem 3.4.2 *Let $p; q; r \in \mathbb{R}_0^+$ such that $p \geq q > 0$. Assume that all the assumptions of **Theorem 3.4.1** are satisfied, furthermore*

$$\int_a^b R(s, m(s))n(s)l(s)\Delta s < 1. \tag{3.16}$$

If the following inequality

$$u^p(t) \leq f(t) + h(t)\int_a^t g(s)u^q(s)\Delta s + \int_a^b w(s)M(s, u^r(s))\Delta s, \tag{3.17}$$

holds, then $u(t)$ has the following estimate

$$u(t) \leq \left(L^*(t) + \frac{M^*(t)\int_a^b w(s)M(s, L^*(s))\Delta s}{1 - \int_a^b R(s, L^*(s))w(s)M^*(s)\Delta s}\right)^{\frac{1}{r}}, \tag{3.18}$$

where

$$\begin{aligned}
L^*(t) &= \frac{r}{p}K^{\frac{r-p}{p}}f(t) + \frac{p-r}{p}K^{\frac{r}{p}} + \frac{r}{p}K^{\frac{r-p}{p}}h(t)\int_a^t Q^*(s)e_{P^*}(t, \sigma(s))\Delta s \\
M^*(t) &= \frac{\frac{r}{p}K^{\frac{r-p}{p}}h(t)e_{P^*}(t, a)}{h(a)}. \qquad (3.19)\\
P^*(t) &= \frac{q}{p}K^{\frac{q-p}{p}}g(t)h(t) \\
Q^*(t) &= \left(\frac{q}{p}K^{\frac{q-p}{p}}f(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right)g(t)
\end{aligned}$$

Proof. Since $h(t)$ is positive and monotonic nondecreasing, we can restate (3.17) as follows

$$u^p(t) \leq f(t) + h(t)\left[\int_a^t g(s)u^q(s)\Delta s + \frac{1}{h(a)}\int_a^b w(s)M(s, u^r(s))\Delta s\right]$$

Denoting by $z(t)$

$$z(t) = \int_a^t g(s)u^q(s)\Delta s + \frac{1}{h(a)}\int_a^b w(s)M(s, u^r(s))\Delta s. \tag{3.20}$$

Clearly $z(t)$ is nonnegative, nondecreasing and applying **Lemma** 1.4.1

$$\begin{aligned} u(t) &\leq \{f(t)+h(t)z(t)\}^{\frac{1}{p}}, \\ u^r(t) &\leq \frac{r}{p}K^{\frac{r-p}{p}}(f(t)+h(t)z(t))+\frac{p-r}{p}K^{\frac{r}{p}}=v(t), \end{aligned} \tag{3.21}$$

and

$$z(a)=\frac{1}{h(a)}\int_a^b w(s)M(s,u^r(s))\Delta s. \tag{3.22}$$

Differentiating (3.20) with respect to t, and then substituting (3.21) into the resulting expression, we obtain :

$$\begin{aligned} z^{\Delta}(t) &= g(t)u^q(t) \\ &\leq g(t)\{f(t)+h(t)z(t)\}^{\frac{q}{p}}. \end{aligned} \tag{3.23}$$

Now, Applying **Lemma** 1.4.1 for (3.23) we get

$$z^{\Delta}(t) \leq \frac{q}{p}K^{\frac{q-p}{p}}g(t)h(t)z(t)+\left(\frac{q}{p}K^{\frac{q-p}{p}}f(t)+\frac{p-q}{p}K^{\frac{q}{p}}\right)g(t). \tag{3.24}$$

The inequality (3.24) can be reformulated as

$$z^{\Delta}(t) \leq P^*(t)z(t)+Q^*(t),$$

where $P^*$and Q^* are defined as in (3.19).

According **Lemma** 1.4.2 , we have

$$z(t) \leq z(a)e_{P^*}(t,a)+\int_a^t Q^*(s)e_{P^*}(t,\sigma(s))\Delta s. \tag{3.25}$$

Substituting (3.22) in (3.25), we obtain

$$\begin{aligned} z(t) &\leq \frac{e_{P^*}(t,a)}{h(a)}\int_a^b w(s)M(s,u^r(s))\Delta s \\ &\quad +\int_a^t Q^*(s)e_{P^*}(t,\sigma(s))\Delta s. \end{aligned} \tag{3.26}$$

Using (3.21) in (3.26), and multipying both side of (3.26) by $\frac{r}{p}K^{\frac{r-p}{p}}h(t)$ and additing the result term $\frac{r}{p}K^{\frac{r-p}{p}}f(t) + \frac{p-r}{p}K^{\frac{r}{p}}$, we obtain

$$\begin{aligned} v(t) &\leq \frac{\frac{r}{p}K^{\frac{r-p}{p}}h(t)e_{P^*}(t,a)}{h(a)}\int_a^b w(s)M(s,v(s))\Delta s + \\ &+\frac{r}{p}K^{\frac{r-p}{p}}f(t) + \frac{p-r}{p}K^{\frac{r}{p}} + \frac{r}{p}K^{\frac{r-p}{p}}h(t)\int_a^t Q^*(s)e_{P^*}(t,\sigma(s))\Delta s. \end{aligned} \tag{3.27}$$

The inequality (3.27) can be reformulated as

$$v(t) \leq L^*(t) + M^*(t)\int_a^b w(s)M(s,v(s))\Delta s \tag{3.28}$$

where $L^*(t)$ and $M^*(t)$ are defined as in (3.19). Applying **Lemma** 3.4.1 to the inequality (3.28), we get

$$v(t) \leq L^*(t) + \frac{M^*(t)\int_a^b w(s)M(s,L^*(s))\Delta s}{1 - \int_a^b R(s,L^*(s))w(s)M^*(s)\Delta s}. \tag{3.29}$$

From (3.21) and (3.29), we get the desired result. ∎

Corollary 3.4.3 *Assume that all the assumptions of* ***Theorem*** 3.4.2 *are satisfied, and let* $\mathbb{T} = \mathbb{Z}$ *. If the following inequality*

$$u(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(\tau)u^q(\tau)\Delta s + \sum_{s=a}^{b-1} w(s)M(s,u^r(s)),$$

holds, then $u(t)$ *has the following estimate*

$$u(t) \leq (L^*(t) + \frac{M^*(t)\sum_{s=a}^{b-1} w(s)M(s,L^*(s))}{1 - \sum_{s=a}^{b-1} R(s,L^*(s))w(s)M^*(s)})^{\frac{1}{r}},$$

where

$$L^*(t) = \frac{r}{p}K^{\frac{r-p}{p}}f(t) + \frac{p-r}{p}K^{\frac{r}{p}} + \frac{r}{p}K^{\frac{r-p}{p}}h(t)\sum_{s=a}^{b-1} Q^*(s)(\prod_{\tau=s+1}^{\tau=1}(1+P^*(\tau))$$

$$M^*(t) = \frac{\frac{r}{p}K^{\frac{r-p}{p}}h(t)(\prod_{\tau=s+1}^{\tau=1}(1+P^*(\tau))}{h(a)}. \tag{3.30}$$

provided

$$\sum_{s=a}^{b-1} R(s, L^*(s))w(s)M^*(s) < 1, \textit{for all } t \in [a; b]_{\mathbb{T}}. \tag{3.31}$$

Theorem 3.4.3 *Let* $u(t)$, $f(t), h(t), g(t), w(t) \in \mathcal{C}_{rd}\left([a; b]_{\mathbb{T}}, \mathbb{R}^+\right)$, *with* $h(t)$ *is a positive and nondecreasing function, and let* $g(t,s)$ *be defined as in* ***Theorem*** 1.1.7 *such that* $g^{\Delta}(t,s) \geq 0$ *for all* $t \geq s$, *let* M *satisfying* (3.1). *If the following inequality*

$$u(t) \leq f(t) + h(t)\int_a^t g(t,s)u(s)\Delta s + \int_a^b w(s)M(s,u(s))\Delta s, \tag{3.32}$$

holds, then $u(t)$ *has the following estimate*

$$u(t) \leq M^*(t) + \frac{L^*(t)\int\limits_a^b w(s)M(s, M^*(s))\Delta s}{1 - \int\limits_a^b R(s, M^*(s))w(s)L^*(s)\Delta s}, \tag{3.33}$$

where

$$\begin{aligned} L(t) &= g(\sigma(t),t)h(t) + \int_a^t g^{\Delta}(t,s)h(s)\Delta s, \\ P(t) &= g(\sigma(t),t)f(t) + \int_a^t g^{\Delta t}(t,s)f(s)\Delta s, \end{aligned} \tag{3.34}$$

and

$$M^*(t) = h(t).P(t) + f(t), L^*(t) = \frac{h(t)e_L(t,a)}{h(a)}. \tag{3.35}$$

Provided

$$\int\limits_a^b R(s, M^*(s))w(s)L^*(s)\Delta s < 1. \tag{3.36}$$

Proof. Since $h(t)$ is positive and monotonic nondecreasing, we can restate (3.32) as follows

$$u(t) \leq f(t) + h(t)\left(\int_a^t g(t,s)u(s)\Delta s + \frac{1}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s\right).$$

Define a function $z(t)$ by

$$z(t) = \int_a^t g(t,s)u(s)\Delta s + \frac{1}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s, \tag{3.37}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq f(t) + h(t)z(t), \tag{3.38}$$

and

$$z(a) = \frac{1}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s. \tag{3.39}$$

Differentiating (3.37) with respect to t, it yields

$$z^{\Delta}(t) = g(\sigma(\ t),t)u(t) + \int_a^t g^{\Delta t}(t,s)u(s)\Delta s \tag{3.40}$$

Substituting (3.38) into (3.40), we get

$$\begin{aligned} z^{\Delta}(t) \ &\leq \ g(\sigma(t),t)f(t) + \int_a^t g^{\Delta t}(t,s)f(s)\Delta s \\ &\quad + g(\sigma(t),t)h(t)z(t) + \int_a^t g^{\Delta t}(t,s)h(s)z(s)\Delta s. \end{aligned} \tag{3.41}$$

Since $z(t)$ is monotonic nondecreasing, we can restate (3.41) as follows

$$z^{\Delta}(t) \leq P(t) + L(t)z(t), \tag{3.42}$$

where $L(t)$ and $P(t)$ are defined as in (3.34).

Applying **Lemma 1.4.2** for (3.42), we obtain

$$z(t) \leq z(a)e_L(t,a) + \int_a^t P(s)e_L(t,\sigma(\tau))\Delta\tau. \tag{3.43}$$

Now, substituting (3.39) in (3.43), then using (3.38) in the resultant inequality, we get

$$z(t) \leq P(t) + \frac{e_L(t,a)}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s, \tag{3.44}$$

Now, multiplying both sides of (3.44) by $h(t)$ and additing the result term $f(t)$, we obtain

$$u(t) \leq h(t).P(t) + f(t) + \frac{h(t)e_L(t,a)}{h(a)}\int_a^b w(s)M(s,u(s))\Delta s \tag{3.45}$$

The inequality (3.45) can be restated as

$$u(t) \leq M^*(t) + L^*(t)\int_a^b w(s)M(s,u(s))\Delta s, \tag{3.46}$$

where $M^*(t)$ and $L^*(t)$ are defined as in (3.35).

Applying **Lemma** 3.4.1, we obtain the desired inequality. ■

Corollary 3.4.4 *Assume that all the assumptions of Theorem* 3.4.3 *are satisfied, and let* $\mathbb{T}=\mathbb{Z}$ *. If the following inequality*

$$u(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(t,\tau)u(\tau) + \sum_{s=a}^{b-1} w(s)M(s,u(s))$$

holds then $u(t)$ *has the following estimate*

$$u(t) \leq M^*(t) + \frac{L^*(t)\sum\limits_{s=a}^{b-1} w(s)M(s,M^*(s))}{1-\sum\limits_{s=a}^{b-1} R(s,M^*(s))w(s)L^*(s)},$$

$$\begin{aligned} M^*(t) &= f(t) + h(t)(g(t+1,t)f(t) + \sum_{s=a}^{t-1}(g(t+1,s)-g(t,s))h(t)f(s), \\ L^*(t) &= \frac{h(t)\prod\limits_{\tau=a}^{\tau=s}[1+L(\tau)]}{h(a)}. \end{aligned}$$

where

Provided $\sum\limits_{s=a}^{b-1} R(s,M^*(s))w(s)L^*(s) < 1.$*and* $L(t) \neq -1$ *for all* $t \in [a;b]_{\mathbb{T}}$.

Theorem 3.4.4 *Under the assumptions of Theorem* 3.4.2*, and let* $p,q,r,K \in \mathbb{R}_0^+$ *such that* $p \geq q > 0; p \geq r > 0$*. Assume*

$$\frac{r}{p}K^{\frac{r-p}{p}}\int_a^b R(s,K^*(s))w(s)h(s)e_Q(s,a)\Delta s < 1. \tag{3.47}$$

If the following inequality

$$u^p(t) \leq f(t) + h(t)\int_a^t g(t,s)u^q(s)\Delta s + \int_a^b w(s)M(s,u^r(s))\Delta s, \tag{3.48}$$

holds, then $u(t)$ has the following estimate

$$u(t) \leq \left\{ K^*(t) + \frac{\frac{r}{p} K^{\frac{r-p}{p}} h(t) e_Q(t,a) \int\limits_a^b w(s) M(s, K^*(s)) \Delta s}{1 - \frac{r}{p} K^{\frac{r-p}{p}} \int\limits_a^b R(s, K^*(s)) w(s) h(s) e_Q(s,a) \Delta s}. \right\}^{\frac{1}{r}}, \tag{3.49}$$

where

$$\begin{aligned} L(t) &= \left(\frac{q}{p} K^{\frac{q-p}{p}} f(t) + \frac{p-q}{p} K^{\frac{q}{p}} \right) g(\sigma(t), t) \\ &\quad + \int_a^t g^{\Delta t}(t,s) \left[\frac{q}{p} K^{\frac{q-p}{p}} f(s) + \frac{p-q}{p} K^{\frac{q}{p}} \right] \Delta s, \end{aligned} \tag{3.50}$$

$$Q(t) = \frac{q}{p} K^{\frac{q-p}{p}} h(t) g(\sigma(t), t) + \frac{q}{p} K^{\frac{q-p}{p}} \int_a^t g^{\Delta t}(t,s) h(s) \Delta s, \tag{3.51}$$

and

$$K^*(t) = \frac{r}{p} K^{\frac{r-p}{p}} h(t) \int_a^t L(\tau) e_Q(t, \sigma(\tau)) \Delta\tau + \frac{r}{p} K^{\frac{r-p}{p}} f(t) + \frac{p-r}{p} K^{\frac{r}{p}}. \tag{3.52}$$

Proof. Since $h(t)$ is positive and monotonic nondecreasing, we can restate (3.48) as follows

$$u^p(t) \leq f(t) + h(t) \left[\int_a^t g(t,s) u^q(s) \Delta s + \int_a^b w(s) M(s, u^r(s)) \Delta s \right].$$

Denoting by $z(t)$

$$z(t) = \int_a^t g(t,s) u^q(s) \Delta s + \int_a^b w(s) M(s, u^r(s)) \Delta s. \tag{3.53}$$

It is clearly $z(t)$ is nonnegative, nondecreasing,

$$\begin{aligned} u(t) &\leq \{f(t) + h(t) z(t)\}^{\frac{1}{p}} \qquad (3.54) \\ u^r(t) &\leq \{f(t) + h(t) z(t)\}^{\frac{r}{p}}, \\ u^r(t) &\leq \frac{r}{p} K^{\frac{r-p}{p}} h(s) z(s) + \frac{r}{p} K^{\frac{r-p}{p}} f(s) + \frac{p-r}{p} K^{\frac{q}{p}} = v(t), \end{aligned}$$

and

$$z(a) = \int_a^b w(s) M(s, u^r(s)) \Delta s. \tag{3.55}$$

Differentiating (3.53)with respect to t, then using (3.54) we get

$$\begin{aligned} z^{\Delta}(t) &= g(\sigma(t),t)u^{q}(t)+\int_{a}^{t} g^{\Delta t}(t,s)u^{q}(s)\Delta s \\ &\leq g(\sigma(t),t)\left(f(t)+h(t)z(t)\right)^{\frac{q}{p}} \\ &\quad +\int_{a}^{t} g^{\Delta t}(t,s)\left(f(s)+h(s)z(s)\right)^{\frac{q}{p}}\Delta s \end{aligned} \tag{3.56}$$

Now, applying **Lemma** 1.4.1 for (3.56) we obtain.

$$\begin{aligned} z^{\Delta}(t) \leq\; & \left(\frac{q}{p}K^{\frac{q-p}{p}}f(t)+\frac{p-q}{p}K^{\frac{q}{p}}\right)g(\sigma(t),t) \\ & +\int_{a}^{t} g^{\Delta t}(t,s)\left(\frac{q}{p}K^{\frac{q-p}{p}}f(s)+\frac{p-q}{p}K^{\frac{q}{p}}\right)\Delta s \\ & +\frac{q}{p}K^{\frac{q-p}{p}}h(t)z(t)g(\sigma(t),t)+\frac{q}{p}K^{\frac{q-p}{p}}\int_{a}^{t} g^{\Delta t}(t,s)h(s)z(s)\Delta s. \end{aligned} \tag{3.57}$$

Since $z(t)$ is monotonic nondecreasing, (3.57) can be restated

$$z^{\Delta}(t) \leq L(t)+Q(t)z(t), \tag{3.58}$$

where $L(t)$ and $Q(t)$ are defined by (3.50) and (3.51) respectively.

According **Lemma** 1.4.2 , inequality (3.58) gives

$$z(t) \leq z(a)e_{Q}(t,a)+\int_{a}^{t} L(\tau)e_{Q}(t,\sigma(\tau))\Delta\tau. \tag{3.59}$$

Substituting (3.55) in (3.59), we obtain

$$z(t) \leq \int_{a}^{t} L(\tau)e_{Q}(t,\sigma(\tau))\Delta\tau+e_{Q}(t,a)\int_{a}^{b} w(s)M(s,u^{r}(s))\Delta s. \tag{3.60}$$

Using (3.54) in (3.60), then applying **Lemma** 1.4.1 for the resultant inequality, we get

$$z(t) \leq \int_{a}^{t} L(\tau)e_{Q}(t,\sigma(\tau))\Delta\tau+e_{Q}(t,a)\int_{a}^{b} w(s)M(s,\frac{r}{p}K^{\frac{r-p}{p}}h(s)z(s)+\frac{r}{p}K^{\frac{r-p}{p}}f(s)+\frac{p-r}{p}K^{\frac{q}{p}})\Delta s, \tag{3.61}$$

Multiplying both sides of (3.61) by $\frac{r}{p}K^{\frac{r-p}{p}}h(t)$, and additing the result term $\frac{r}{p}K^{\frac{r-p}{p}}f(t)+\frac{p-r}{p}K^{\frac{r}{p}}$,we obtain

$$v(t) \leq K^{*}(t)+\frac{r}{p}K^{\frac{r-p}{p}}h(t)e_{Q}(t,a)\int_{a}^{b} w(s)M(s,v(s))\Delta s, \tag{3.62}$$

where $K^*(t)$ is defined as in (3.52).Applying **Lemma** 3.4.1 to inequality (3.62),we obtain

$$v(t) \leq K^*(t) + \frac{\frac{r}{p}K^{\frac{r-p}{p}} h(t) e_Q(t,a) \int_a^b w(s) M(s, K^*(s)) \Delta s}{1 - \frac{r}{p}K^{\frac{r-p}{p}} \int_a^b R(s, K^*(s)) w(s) h(s) e_Q(s,a) \Delta s}. \tag{3.63}$$

Combining (3.63), (3.61), and (3.54), we get the desired result. ■

Corollary 3.4.5 *Assume that all the assumptions of Theorem* 3.4.3 *are satisfied, and let* $\mathbb{T} = \mathbb{Z}$ *. If the following inequality*

$$u^p(t) \leq f(t) + h(t) \sum_{\tau=a}^{t-1} g(t,\tau) u^q(\tau) + \sum_{s=a}^{b-1} w(s) M(s, u^r(s)),$$

holds then $u(t)$ *has the following estimate*

$$u(t) \leq \left\{ K^*(t) + \frac{\frac{r}{p}K^{\frac{r-p}{p}} h(t) \prod_{\tau=s+1}^{\tau=t} [1 + Q(\tau)] \sum_{s=a}^{b-1} w(s) M(s, K^*(s))}{1 - \frac{r}{p}K^{\frac{r-p}{p}} \sum_{s=a}^{b-1} R(s, K^*(s)) w(s) h(s) \prod_{\tau=s+1}^{\tau=t} [1 + Q(\tau)]} \right\}^{\frac{1}{p}}$$

where

$$\begin{aligned} L(t) &= \left(\frac{q}{p} K^{\frac{q-p}{p}} f(s) + \frac{p-q}{p} K^{\frac{q}{p}} \right) g(\sigma(t), t) \\ &\quad + \sum_{s=a}^{t-1} (g(t+1, s) - g(t,s)) \left[\frac{q}{p} K^{\frac{q-p}{p}} f(s) + \frac{p-q}{p} K^{\frac{q}{p}} \right], \end{aligned}$$

$$Q(t) = \frac{q}{p} K^{\frac{q-p}{p}} h(t) g(t+1, t) + \frac{q}{p} K^{\frac{q-p}{p}} \sum_{s=a}^{t-1} (g(t+1,s) - g(t,s)) h(s),$$

and

$$K^*(t) = \frac{r}{p} K^{\frac{r-p}{p}} h(t) \sum_{s=a}^{t-1} L(\tau) \prod\nolimits_{\tau=a}^{\tau=s} [1 + Q(\tau)] + \frac{r}{p} K^{\frac{r-p}{p}} f(t) + \frac{p-r}{p} K^{\frac{r}{p}}. \tag{3.64}$$

.

Provided $\frac{r}{p}K^{\frac{r-p}{p}} \sum_{s=a}^{b-1} R(s, K^*(s)) w(s) h(s) \prod_{\tau=s+1}^{\tau=t} [1 + Q(\tau)] < 1$ *, and* $Q(\tau) \neq -1$ *for all* $t \in [a; b]_{\mathbb{T}}$.

3.4.1 Applications

In this section, we present an application of Theorem 3.4.2 to derive an explicit bound for the solution of a certain differential equation on time scale.

Consider the following general mixed nonlinear integral equation

$$y^p(t) = x(t) + h(t)\int_\alpha^\eta F(s, y^q(s))\Delta s + \int_\alpha^\beta G(s, y^r(s))\Delta s, \tag{3.65}$$

where $p > 1$ is a constant, y, $x : \mathbb{T} \to \mathbb{R}$ are right-dense continuous functions on $\mathbb{T}$, F $:\mathbb{T} \times\mathbb{T} \times\mathbb{R} \to \mathbb{R}$ is right-dense continuous function on $\mathbb{T} \times\mathbb{T}$ and continuous on $\mathbb{R}$, and G: $\mathbb{T}\times \mathbb{R} \to \mathbb{R}$ is right-dense continuous function on $\mathbb{T}$ and continuous on $\mathbb{R}$.

Proposition 3.4.1 *Assume that*

$$\begin{aligned} |F(t,s,u(s))| &\leq g(s)\,|u(s)|^q \\ |G(s,u(s))| &\leq w(s)M(s,|u(s)|^r), \end{aligned} \tag{3.66}$$

where M is defined as in (3.1),*and h, g, w and q, r satisfy the hypotheses of Theorem* 3.4.2. *If $y(t)$ is any solution of* (3.65) – (3.66), *then $y(t)$ satisfies the following estimate*

$$|y(t)| \leq \left\{ L^*(t) + \frac{M^*(t)\int\limits_a^b w(s)M(s, M^*(s))\Delta s}{1 - \int\limits_a^b R(s, M^*(s))w(s)(s)\Delta s} \right\}^{\frac{1}{p}}, \tag{3.67}$$

where $L^*(t)$ and $M^*(t)$ are defined by (3.19).

Proof. Let $y(t)$ be a solution of (3.65), using the properties of modulus, we obtain

$$|y(t)|^p \leq |x(t)| + h(t)\int_a^t |F(t,s,x(s))|\,\Delta s + \int_a^b |G(s,x(s))|\,\Delta s. \tag{3.68}$$

Using (3.66) and the assumptions imposed on f, g, h and w, we can restate (3.68) as follows

$$|y(t)|^p \leq |x(t)| + h(t)\int_a^t g(s)\,|u(s)|^q\,\Delta s + \int_a^b w(s)M(s, |u(s)^r|)\Delta s. \tag{3.69}$$

Now, an application of Theorem 3.4.2 for (3.69) gives the estimate (3.67). ■

CHAPTRE 4

Practical h-Stability Results for Solutions to Certain Nonlinear Perturbed Systems on Time Scales and Their Applications

4.1 Introduction

Perturbation theory is a relevant field for applications in time scale dynamics, which comprises a set of systematic methods used to evaluate the global behavior of solutions to dynamical systems defined on non-uniform domains.In the study of the stability properties of differential and difference systems, a new concept called h-stability was introduced by Pinto in [97] .This form of stability is stronger than exponential stability.

In the first section, we present the statement of the problem by providing the general form of the solution.

In the second section, we discuss some generalizations that have appeared in the literature and demonstrated their significance in various areas of dynamical systems.

In the final section, we establish new results concerning the global practical uniform h-stability of time-varying nonlinear perturbed systems. These results are compiled in an article submitted for possible publication in an international journal.

4.2 Statement of Problem

In this chapter, we investigate the **global practical uniform h-stability** of time-varying nonlinear perturbed systems of the form

$$x^{\triangle}(t) = A(t)x + B(t)x + F(t,x). \tag{4.1}$$

The solution of the perturbed system (4.1) denoted $x(t) = x(t, t_0, x_0)$ verifies the integral equation

$$x(t) = \phi_A(t,t_0)x_0 + \int_{t_0}^{t} \phi_A(t,\sigma(s))\left(B(s)x(s) + F(s,x(s))\right) \triangle s \qquad t \in \mathbb{T}^{+}_{t_0}.$$

Recall that $\phi_A(t,t_0)$ is a matrix function and is called a transition matrix (*see Definition* 1.3.6).

Where $A(.), B(.) \in C_{rd}(\mathbb{T}, M_n(\mathbb{R}))$ and $F : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in time and locally Lipschitz in x, uniformly in t. The system (4.1) is seen as a perturbation of the nominal system:

$$x^{\triangle}(t) = A(t)x, \; x(t_0) = x_0, \; x_0 \neq 0 \tag{4.2}$$

The goal is to represent the perturbation as an additive term on the right-hand side of the state equation. The perturbation terms $B(t)$ and $F(t,x)$ could result from errors in modelling the nonlinear system, ageing of parameters, or uncertainties and disturbances which exist in any realistic problem. We will direct our special attention to the perturbed term $F(t,0)$. In a typical situation, we do not know $F(t,x)$, but we know some information about it, like knowing an upper bound on $\|F(t,x)\|$. The trivial question posed here is: If the linear nominal system presents one of the stability types, the perturbed one keeps the same behavior or not. In this study, we give a condition on $B(t)$ and we introduce two cases on $F(t,x)$. The rst one is when $F(t,0) = 0$, while the second case is when $F(t,0) \neq 0$.

In the first section of this chapter, we first cite some results of **asymptotic stability** of the perturbed system (4.1) with $B = 0$ presented by **B. B. Nasser, K. Boukerrioua, D. Diabi, M. Meramria and M.A.Hammami** [17, 18, 35] . In a second section, we present some new results on the study **h-satability** and **h-practical stability** of non linear perturbed system (4.2), under different conditions imposed on the perturbation F and B and we conclude this chapter by an illustrative example.

We note that these new results were written in an article submitted for possible publication in an indexed international journal.

4.3 On some recent results around exponential stability

In this section, we present without proof some results on the stability of the system (4.2) with $B = 0$ obtained in [17, 18, 35].

Theorem 4.3.1 ([17]) *If the following conditions are satisfied*

- the linear system (4.2) is UES.
- The term perturbed satisfies

$$F(t,x) \leq \alpha(t)\|x\| + \chi(t),$$

- $\int_{t_0}^{+\infty} \frac{\alpha(s)}{1-\lambda\mu(s)}\Delta s \leq \widetilde{d} < +\infty, \; \int_{t_0}^{+\infty} \chi(s) e_{-\lambda}(t_0, \sigma(s))\,\Delta s \leq \overline{k} < +\infty.$

Then the perturbed system (4.1) with $B = 0$ is uniformly exponentially stable

Theorem 4.3.2 ([18]) *If the following conditions are satisfied*

- the linear system (4.2) is uniformly exponentially stable.
- The term perturbed satisfies

$$\|F(t,x)\| \leq m(t)\, \|x\|^p\,,$$

where $m \in C_{rd}(\mathbb{T}^+_{t_0}, \mathbb{R}_+)$ and $p \in]0,1[$

- $\int_{t_0}^{\infty} e_{-\lambda}(a,\sigma(s))m(s))^{\frac{1}{q}}\Delta s \leq \widetilde{m}$ with $q = 1-p$.

Then the perturbed system (4.1) with $B = 0$ is uniformly exponentially stable.

Theorem 4.3.3 ([*35*]) *If the following conditions are satisfied*

■ the linear system (4.2) is UES.

■ The term perturbed satisfies

$$F(t,x) \leq g(t)w(\|x\|), t \in t \in \mathbb{T}^+_{t_0}.$$

Where $g\,(t)$ is a positive and $rd-$continuous function , $w \in \widehat{H}$. Let r be the solution of

$$r^{\Delta}(t) = p(t)w(r(t)), r(t_0) = \gamma$$

■ There exists a bijective function W satisfying

$$\begin{aligned} (W \circ r)^{\Delta} &= \gamma p \text{ with} \int_{t_0}^{\infty} p(s)\Delta s < \infty, \\ p(t) &= \frac{\gamma e_{-\lambda}(t_0,\sigma(t))}{\|x_0\|} g(t) w\left(\frac{\|x_0\|}{\gamma e_{-\lambda}(t_0,t)}\right). \end{aligned}$$

Then the perturbed system (4.1) with $B = 0$ is uniformly exponentially stable.

Theorem 4.3.4 ([*35*]) *If the following conditions are satisfied*

■ the linear system (4.2) is UES.

- The term perturbed satisfies the condition:

$$\|F(t,x)\| \leq \eta\left(d(t)\|x\| + k(t)\right),$$

where $d, k \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$, $\eta\ : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $]0,\infty[$such that its first derivative η' is continuous and decreasing on $]0,\infty[$.

- There are two constants $\widetilde{d}, \widetilde{k} \geq 0$ such that

$$\int_{t_0}^{+\infty} \frac{\eta'(k(s))d(s)}{1-\lambda\mu(s)}\Delta s \leq \widetilde{d} < +\infty, \quad \int_{t_0}^{+\infty} \eta(k(s))e_{-\lambda}(t_0,\sigma(s))\Delta s \leq \widetilde{k} < +\infty.$$

Then the perturbed system (4.1) with $B = 0$ is uniformly exponentially stable.

Theorem 4.3.5 ([*35*]) *If the following conditions are satisfied*

- the linear system (4.2) is UES.
- The term perturbed satisfies the the following relation:

$$F(t,x) \leq g(t)w(\|x\|), t \in \mathbb{T}_{t_0}^+,$$

Where $g(t)$ a positive and $rd-$continuous function, $w \in \mathcal{F}$. Let r be the solution of

$$r^{\Delta}(t) = p_1(t)w(r(t)), r(t_0) = \gamma$$

- There exists a bijective function G satisfying

$$(W \circ r)^{\Delta} = p_1 \text{ with} \int_{t_0}^{\infty} p_1(s)\Delta s < \infty \text{ and } p_1(t) = \frac{\gamma g(t)}{1-\lambda\mu(t)}$$

Then the perturbed system (4.1) with $B = 0$ is uniformly exponentially stable.

4.4 New results on the study of h-satability and practical h-stablity

We now state some new results on the h-stability of the perturbed systems (4.1) under various conditions imposed on the perturbation term F and B.

Theorem 4.4.1 *If the following conditions are satisfied*

i) the linear system (4.2) is globally uniformly h-stable.

ii) The term perturbed satisfies

$$\begin{aligned} \|F(t,x)\| &\leq \eta(t)w(\|x(t)\|), \qquad t \in \mathbb{T}, \\ \|B(t)\| &\leq \gamma(t) \end{aligned} \tag{4.3}$$

and

$$w(\|x(t)\|) \leq M \|x(t)\| \tag{4.4}$$

where η, w and γ are nonnegative $rd-$continuous functions on $\mathbb{T}_{t_0}^{+}$

iii) There exists a constant N such that

$$\int_{t_0}^{t} \left(\frac{h(s)}{h(\sigma(s))} (\gamma(s) + M\eta(s)) \right) \triangle s \leq N, \tag{4.5}$$

Then the perturbed system (4.1) is globally uniformly h-stable.

Proof. Let $x(t)$ be the solution of system (4.1), then

$$x(t) = \phi_A(t,t_0)x_0 + \int_{t_0}^{t} \phi_A(t,\sigma(s)) \left(B(s)x(s) + F(s,x(s))\right) \triangle s, \tag{4.6}$$

where $\phi_A(t,t_0)$ is the solution of the linear system (4.2). Thus, from the global uniform h-stability of system (4.2) and according to condition $ii)$, we obtain

$$\begin{aligned} \|x(t)\| &\leq \frac{ch(t)}{h(t_0)} \|x_0\| + ch(t) \int_{t_0}^{t} \frac{1}{h(\sigma(s))} \left(\|B(s)\| \|x(s)\| + \|F(s,x(s))\|\right) \triangle s \\ &\leq \frac{ch(t)}{h(t_0)} \|x_0\| + ch(t) \int_{t_0}^{t} \frac{1}{h(\sigma(s))} \left(\gamma(s) \|x(s)\| + \eta(s)w(\|x(s)\|)\right) \triangle s \\ &\leq c\frac{\|x_0\|}{h(t_0)}h(t) + ch(t) \int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))} \left(\gamma(s) + M\eta(s)\right) \frac{\|x(s)\|}{h(s)} \triangle s. \end{aligned} \tag{4.7}$$

Now let us set , $\vartheta(t) = \frac{\|x(t)\|}{h(t)}$, we obtain

$$\vartheta(t) \leq c\vartheta(t_0) + c\int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))} \left(\gamma(s) + M\eta(s)\right) \vartheta(s) \triangle s. \tag{4.8}$$

By applying *Lemma* 1.4.3, we have for all $t \in \mathbb{T}_{t_0}^{+}$

$$\begin{aligned} \vartheta(t) &\leq c\vartheta(t_0) + c\int_{t_0}^{t}\left(c\vartheta(t_0)\frac{h(s)}{h(\sigma(s))}\left(\gamma(s)+M\eta(s)\right)\right) \\ &\quad \exp\left(c\int_{\sigma(s)}^{s}\frac{h(\tau)}{h(\sigma(\tau))}\left(\gamma(\tau)+M\eta(\tau)\right)\triangle\tau\right)\triangle s. \end{aligned} \tag{4.9}$$

From condition $iii)$,we have

$$\vartheta(t) \leq c\left(1+cNe^{cN}\right)\vartheta(t_0), \tag{4.10}$$

we deduce that, for all $t \in \mathbb{T}_{t_0}^{+}$and $x_0 \in \mathbb{R}^n$ that, the solution of system (4.1) satisffies

$$\|x(t)\| \leq \frac{k\,\|x_0\|\,h(t)}{h(t_0)}, \tag{4.11}$$

where

$$k = c\left(1+cNe^{cN}\right) \geq 1$$

Then, the perturbed system (4.1) is globally uniformly h-stable. ■

Theorem 4.4.2 *If the following conditions are satisfied*

i) the linear system (4.2) is globally uniformly h-stable.

ii) The term perturbed satisfies

$$\begin{aligned} \|F(t,x)\| &\leq h(t)p(t), \qquad t \in \mathbb{T}, \\ \|B(t)\| &\leq \gamma(t) \end{aligned} \tag{4.12}$$

where p and γ are non-negative $rd-$continuous functions on $\mathbb{T}_{t_0}^{+}$

iii) There are two constants N_1 , N_2such that

$$\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}\gamma(s)\triangle s \leq N_1, \int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}p(s)\triangle s \leq N_2 \tag{4.13}$$

Then the perturbed system (4.1) is globally practically uniformly h-stable.

Proof. Let $x(t)$ be the solution of system (4.1), then

$$x(t) = \phi_A(t,t_0)x_0 + \int_{t_0}^{t} \phi_A(t,\sigma(s))\left(B(s)x(s) + F(s,x(s))\right) \triangle s, \tag{4.14}$$

where $\phi_A(t,t_0)$ is the solution of the linear system (4.2). Thus, from the global uniform h-stability of system (4.2) and according to condition ii), we obtain

$$\begin{aligned} \|x(t)\| &\leq \frac{ch(t)}{h(t_0)}\|x_0\| + ch(t)\int_{t_0}^{t}\frac{1}{h(\sigma(s))}\left(\|B(s)\|\|x(s)\| + \|F(s,x(s))\|\right)\triangle s \quad (4.15) \\ &\leq \frac{ch(t)}{h(t_0)}\|x_0\| + ch(t)\int_{t_0}^{t}\frac{1}{h(\sigma(s))}\left(\gamma(s)\|x(s)\| + h(s)p(s)\right)\triangle s \\ &\leq \frac{ch(t)}{h(t_0)}\|x_0\| + ch(t)\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}\left(\gamma(s)\frac{\|x(s)\|}{h(s)} + p(s)\right)\triangle s \end{aligned}$$

Now let us set , $\vartheta(t) = \frac{\|x(t)\|}{h(t)}$ we get

$$\vartheta(t) \leq c\vartheta(t_0) + c\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}\gamma(s)\vartheta(s) + \frac{h(s)}{h(\sigma(s))}p(s)\triangle s. \tag{4.16}$$

By applying *Lemma* 1.4.4, we obtain for all $t \in \mathbb{T}_{t_0}^{+}$

$$\begin{aligned} \vartheta(t) \leq\ & c\vartheta(t_0) + c\int_{t_0}^{t}\left(c\frac{h(s)}{h(\sigma(s))}\gamma(s)\vartheta(t_0) + \frac{h(s)}{h(\sigma(s))}p(s)\right) \\ & \exp\left(c\int_{\sigma(s)}^{s}\frac{h(\tau)}{h(\sigma(\tau))}\gamma(\tau)\triangle\tau\right)\triangle s. \end{aligned} \tag{4.17}$$

From conditions iii),we have

$$\vartheta(t) \leq c\left(1 + cN_1e^{cN_1}\right)\vartheta(t_0) + cN_2e^{cN_1}, \tag{4.18}$$

we deduce that, for all $t \in \mathbb{T}_{t_0}^{+}$ and all $x_0 \in \mathbb{R}^n$, that, the solution of system (4.1) satisffies

$$\|x(t)\| \leq \frac{k\|x_0\|h(t)}{h(t_0)} + \rho, \tag{4.19}$$

where

$$k = c\left(1 + cN_1e^{cN_1}\right) \geq 1 \text{ and } \rho = cN_2e^{cN_1}\|h\|_{\infty} > 0$$

Then the perturbed system (4.1) is globally practically uniformly h-stable. ■

Theorem 4.4.3 *If the following conditions are satisfied*

i) the linear system (4.2) is globally uniformly h-stable.

ii) The term perturbed satisfies

$$\begin{aligned} \|F(t,x)\| &\leq P(t, \|x(t)\|) \quad P(t,0)=0, \\ \|B(t)\| &\leq \gamma(t), \end{aligned} \tag{4.20}$$

where $P : \mathbb{T}_{t_0}^{+} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a rd-continuously function such that

$$0 \leq P(t,x) - P(t,y) \leq L(t,y)(x-y), \quad x \geq y \geq 0, \tag{4.21}$$

iii) $L : \mathbb{T}_{t_0}^{+} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a rd-continuously function, γ is non-negative rd-continuous function on $\mathbb{T}_{t_0}^{+}$ satisfying

$$\int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))} \left(\gamma(s) + L(s,0)\right) \triangle s \leq N, \tag{4.22}$$

Then the perturbed system (4.1) is globally uniformly h-stable.

Proof. Let $x(t)$ be the solution of system (4.1), then

$$x(t) = \phi_A(t,t_0)x_0 + \int_{t_0}^{t} \phi_A(t,\sigma(s)) \left(B(s)x(s) + F(s,x(s))\right) \triangle s, \tag{4.23}$$

where $\phi_A(t,t_0)$ is the solution of the linear system (4.2). Thus, from the global uniform h-stability of system (4.2) and according to condition ii), we obtain

$$\begin{aligned} \|x(t)\| &\leq \frac{ch(t)}{h(t_0)} \|x_0\| + ch(t) \int_{t_0}^{t} \frac{1}{h(\sigma(s))} \left(\|B(s)\| \, \|x(s)\| + \|F(s,x(s))\|\right) \triangle s \\ &\leq \frac{ch(t)}{h(t_0)} \|x_0\| + ch(t) \int_{t_0}^{t} \frac{1}{h(\sigma(s))} (\gamma(s) \|x(s)\| + P(s, \|x(s)\|)) \triangle s \\ &\leq \frac{ch(t)}{h(t_0)} \|x_0\| + ch(t) \int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))} \left(\gamma(s) \frac{\|x(s)\|}{h(s)} + L(s,0) \frac{\|x(s)\|}{h(s)}\right) \triangle s. \end{aligned} \tag{4.25}$$

Now let us say, $\vartheta(t) = \frac{\|x(t)\|}{h(t)}$, we get

$$\vartheta(t) \leq c\vartheta(t_0) + c \int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))} \left(\gamma(s) + L(s,0)\right) \vartheta(s) \triangle s. \tag{4.26}$$

By applying *Lemma* 1.4.3, we obtain for all $t \in \mathbb{T}_{t_0}^{+}$

$$\begin{aligned} \vartheta(t) \ &\leq \ c\vartheta(t_0) + c\int_{t_0}^{t} \left(c\vartheta(t_0)\frac{h(s)}{h(\sigma(s))}\left(\gamma(s) + L(s,0)\right)\right) \\ &\quad \exp\left(c\int_{\sigma(s)}^{s} \frac{h(\tau)}{h(\sigma(\tau))}\left(\gamma(\tau) + L(\tau,0)\right) \triangle\, \tau\right) \triangle\, s. \end{aligned} \tag{4.27}$$

From condition *iii)*,we have

$$\vartheta(t) \leq c\left(1 + cNe^{cN}\right)\vartheta(t_0), \tag{4.28}$$

we deduce that, for all $t \in \mathbb{T}_{t_0}^{+}$ and all $x_0 \in \mathbb{R}^n$ that, the solution of system (4.1) satisffies

$$\|x(t)\| \leq \frac{k\|x_0\| h(t)}{h(t_0)}, \tag{4.29}$$

where

$$k = c\left(1 + cNe^{cN}\right) \geq 1$$

Then the perturbed system (4.1) is globally uniformly h-stable. ■

Theorem 4.4.4 *If the following conditions are satisfied*

i) the linear system (4.2) is globally uniformly h-stable.

ii) The term perturbed satisfies

$$\begin{aligned} \|B(t)\| \ &\leq \ \gamma(t), \\ \|F(t,x)\| \ &\leq \ l(t)\|x(t)\| + v(t) \end{aligned} \tag{4.30}$$

where l, v and γ are non-negative $rd-$continuous functions on $\mathbb{T}_{t_0}^{+}$

iii) There are two constants N_1 , N_2such that

$$\int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))}\left(\gamma(s) + l(s)\right) \triangle\, s \leq N_1, \int_{t_0}^{t} \frac{v(s)}{h(\sigma(s))} \triangle\, s \leq N_2 \tag{4.31}$$

Then the perturbed system (4.1) is globally practically uniformly h-stable.

Proof. Let $x(t)$ be the solution of system (4.1), then

$$x(t) = \phi_A(t,t_0)x_0 + \int_{t_0}^{t} \phi_A(t,\sigma(s))\left(B(s)x(s) + F(s,x(s))\right) \triangle s \tag{4.32}$$

where $\phi_A(t,t_0)$ is the solution of the linear system (4.2). Thus, from the global uniform h-stability of system (4.2) and according to condition ii), we obtain

$$\begin{aligned} \|x(t)\| &\leq \frac{ch(t)}{h(t_0)}\|x_0\| + ch(t)\int_{t_0}^{t}\frac{1}{h(\sigma(s))}\left(\|B(s)\|\,\|x(s)\| + \|F(s,x(s))\|\right)\triangle s \\ &\leq \frac{ch(t)}{h(t_0)}\|x_0\| + ch(t)\int_{t_0}^{t}\frac{1}{h(\sigma(s))}\left(\gamma(s)\|x(s)\| + l(t)\|x(s)\| + v(s)\right)\triangle s \\ &\leq \frac{ch(t)}{h(t_0)}\|x_0\| + ch(t)\int_{t_0}^{t}\frac{1}{h(\sigma(s))}\left(h(s)\left(\gamma(s)+l(s)\right)\frac{\|x(s)\|}{h(s)} + v(s)\right)\triangle s \end{aligned} \tag{4.33}$$

Now let us say, $\vartheta(t) = \frac{\|x(t)\|}{h(t)}$ we get

$$\vartheta(t) \leq c\vartheta(t_0) + c\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}\left(\gamma(s)+l(s)\right)\vartheta(s) + \frac{v(s)}{h(\sigma(s))}\triangle s. \tag{4.34}$$

By applying *Lemma* 1.4.4, we obtain for all $t \in \mathbb{T}_{t_0}^{+}$ that

$$\begin{aligned} \vartheta(t) \quad \leq \quad & c\vartheta(t_0) + c\int_{t_0}^{t}\left(c\frac{h(s)}{h(\sigma(s))}\left(\gamma(s)+l(s)\right)\vartheta(t_0) + \frac{v(s)}{h(\sigma(s))}\right) \\ & \exp\left(c\int_{\sigma(s)}^{s}\frac{h(\tau)}{h(\sigma(\tau))}\left(\gamma(\tau)+l(\tau)\right)\triangle\tau\right)\triangle s. \end{aligned} \tag{4.35}$$

From conditions iii),we have

$$\vartheta(t) \leq c\left(1 + cN_1e^{cN_1}\right)\vartheta(t_0) + cN_2e^{cN_1}, \tag{4.36}$$

we deduce that, for all $t \in \mathbb{T}_{t_0}^{+}$ and all $x_0 \in \mathbb{R}^n$ that, the solution of system (4.1) satisffies

$$\|x(t)\| \leq \frac{k\|x_0\|h(t)}{h(t_0)} + \rho, \tag{4.37}$$

where

$$k = c\left(1 + cN_1e^{cN_1}\right) \geq 1 \text{ and } \rho = cN_2e^{cN_1}\|h\|_\infty > 0. \tag{4.38}$$

Then the perturbed system (4.1) is globally practically uniformly h-stable. ■

4.5 Examples

In this section, we introduce some examples to illustrate the effectiveness of the obtained results.

Example 4.5.1 *Let $\mathbb{T}$ be a mixed continuous-discrete time scale and $t_0 = 0$, The discrete part has non-uniform step size. The graininess function is bounded as follows:*

$$0 \le \mu(t) \le \mu_{\max} = \frac{1}{2}, \forall t \in \mathbb{T}_0^+.$$

Consider the following time-varying system:

$$\begin{cases} x_1^{\triangle}(t) = -x_1(t) + \frac{1}{\sqrt{2}} \frac{\sigma(t)+t+t\sigma(t)}{(t+2)^2(\sigma(t)+2)^2} x_1(t) + \frac{1}{\sqrt{2}} \frac{t+\sigma(t)+10}{(t+5)^2(\sigma(t)+5)^2} \frac{x_1(t)}{\sqrt{x_1^2(t)+x_2^2(t)}} \\ x_2^{\triangle}(t) = -x_2(t) + \frac{1}{\sqrt{2}} \frac{\sigma(t)+t+t\sigma(t)}{(t+2)^2(\sigma(t)+2)^2} x_2(t) + \frac{1}{\sqrt{2}} \frac{t+\sigma(t)+10}{(t+5)^2(\sigma(t)+5)^2} \frac{x_2(t)}{\sqrt{x_1^2(t)+x_2^2(t)}}. \end{cases} \tag{4.39}$$

where $x(t_0) = x_0$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in C_{rd}(\mathbb{T}, M_2(\mathbb{R}))$, $B(t) \in C_{rd}(\mathbb{T}, M_2(\mathbb{R}))$ *and* $f :$ $\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$f(t,x) = \frac{1}{\sqrt{2}\,\|x(t)\|} \frac{t+\sigma(t)+10}{(t+5)^2(\sigma(t)+5)^2} x(t)$$

$$\phi_A(t,t_0) = \begin{pmatrix} e_{-1}(t,t_0) & 0 \\ 0 & e_{-1}(t,t_0) \end{pmatrix}, \tag{4.40}$$

therefore,

$$\|\phi_A(t,t_0)\| = \sqrt{2} e_{-1}(t,t_0) \tag{4.41}$$

by taking $c \ge \sqrt{2}$ and $h(t) = e_{-1}(t,0)$, we get

$$\|\phi_A(t,t_0)\| \le \sqrt{2} e_{-1}(t,0) e_{-1}(0,t_0) = c h(t) h(t_0)^{-1} \tag{4.42}$$

which ensures the h-stability of the homogenous system of the form $x^{\triangle}(t) = Ax(t)$. Furthermore, we have

$$h(\sigma(t)) = e_{-1}(\sigma(t),0) = (1-\mu(t))\, e_{-1}(t,0). \tag{4.43}$$

Let now verify the others two conditions of Theorem 4.4.2. *It's clear that*

$$\begin{aligned} \|f(t,x(t))\| &\leq \frac{t+\sigma(t)+10}{(t+5)^2(\sigma(t)+5)^2} \\ \|B(t)\| &\leq \gamma(t)=\frac{\sigma(t)+t+t\sigma(t)}{(t+2)^2(\sigma(t)+2)^2}, \end{aligned} \tag{4.44}$$

also

$$\begin{aligned} \int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}\gamma(s)\triangle s &\leq \int_0^{+\infty}\frac{e_{-1}(s,0)}{(1-\mu(s))e_{-1}(s,0)}\frac{\sigma(s)+s+s\sigma(s)}{(s+2)^2(\sigma(s)+2)^2}\triangle s \\ &= \int_0^{+\infty}\frac{\sigma(s)+s+s\sigma(s)}{(1-\mu(s))(s+2)^2(\sigma(s)+2)^2} \\ &\leq 2\int_0^{+\infty}\frac{\sigma(s)+s+s\sigma(s)}{(s+2)^2(\sigma(s)+2)^2}\triangle s \\ &= 2\int_0^{+\infty}\left(\frac{-(s+1)}{(s+2)^2}\right)^{\triangle}\triangle s \\ &= \frac{1}{2}<\infty \end{aligned} \tag{4.45}$$

$$\begin{aligned} \int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}p(s)\triangle s &= \int_0^{+\infty}\frac{e_{-1}(s,0)}{(1-\mu(s))e_{-1}(s,0)}\frac{s+\sigma(s)+10}{(s+5)^2(\sigma(s)+5)^2}\triangle s \\ &\leq 2\int_0^{+\infty}\frac{s+\sigma(s)+10}{(s+5)^2(\sigma(s)+5)^2}\triangle s \\ &= 2\int_0^{+\infty}\left(\frac{-1}{(s+5)^2}\right)^{\triangle}\triangle s \\ &= \frac{2}{25}<+\infty \end{aligned} \tag{4.46}$$

All statements of Theorem 4.4.2 *are approved. So, we deduce that the systeme* (4.39) *is globally practically uniformly h-stable.*

Example 4.5.2 *Consider the following time-varying system:*

$$\begin{cases} x_1^{\triangle}(t)=-\frac{1}{2}x_1(t)+\frac{1}{\sqrt{2}}\frac{t\sigma(t)}{(t^3+1)(\sigma^3(t)+1)}x_1(t)+\frac{1}{\sqrt{2}}\frac{t^2+\sigma^2(t)}{(t^3+1)(\sigma^3(t)+1)}x_1(t)+\frac{1}{\sqrt{2}}\frac{2t+2\sigma(t)}{(t^2+1)(\sigma^2(t)+1)}e_{-\frac{1}{2}}(t,0) \\ x_2^{\triangle}(t)=-\frac{1}{2}x_2(t)+\frac{1}{\sqrt{2}}\frac{t\sigma(t)}{(t^3+1)(\sigma^3(t)+1)}x_2(t)+\frac{1}{\sqrt{2}}\frac{t^2+\sigma^2(t)}{(t^3+1)(\sigma^3(t)+1)}x_2(t)+\frac{1}{\sqrt{2}}\frac{2t+2\sigma(t)}{(t^2+1)(\sigma^2(t)+1)}e_{-\frac{1}{2}}(t,0) \end{cases} \tag{4.47}$$

where $x(t)=(x_1(t),x_2(t))^T\in\mathbb{R}^2$, $A=\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2}\end{pmatrix}\in C_{rd}(\mathbb{T},M_2(\mathbb{R}))$, $B(t)\in C_{rd}(\mathbb{T},M_2(\mathbb{R}))$ *and* $f:\mathbb{T}\times\mathbb{R}^2\to\mathbb{R}^2$ *such that*

$$f(t,x)=\frac{1}{\sqrt{2}}\frac{t\sigma(t)}{(t^3+1)(\sigma^3(t)+1)}x(t)+\frac{1}{\sqrt{2}}(\frac{2t+2\sigma(t)}{(t^2+1)(\sigma^2(t)+1)}e_{-\frac{1}{2}}(t,0)$$

$$\phi_A(t,t_0)=\begin{pmatrix} e_{-\frac{1}{2}}(t,t_0) & 0 \\ 0 & e_{-\frac{1}{2}}(t,t_0)\end{pmatrix} \tag{4.48}$$

therefore,

$$\|\phi_A(t,t_0)\|=\sqrt{2}e_{-\frac{1}{2}}(t,t_0) \tag{4.49}$$

by taking $c\geq\sqrt{2}$ *and* $h(t)=e_{-\frac{1}{2}}(t,0)$, *we get*

$$\|\phi_A(t,t_0)\|\leq\sqrt{2}e_{-\frac{1}{2}}(t,0)e_{-\frac{1}{2}}(0,t_0)=ch(t)h(t_0)^{-1} \tag{4.50}$$

which ensures the h-stability of the homogenous equation of the form $x^{\triangle}(t)=Ax(t)$. *Furthermore, we have*

$$h(\sigma(t))=e_{-\frac{1}{2}}(\sigma(t),0)=\left(1-\frac{1}{2}\mu(t)\right)e_{-\frac{1}{2}}(t,0) \tag{4.51}$$

Let now verify the others two conditions of Theorem 4.4.4. *It's clear that*

$$\begin{aligned}\|f(t,x(t))\| &\leq l(t)\|x(t)\|+v(t)\\ &= \frac{t^2+\sigma^2(t)}{(t^3+1)(\sigma^3(t)+1)}\|x(t)\|+\frac{2t+2\sigma(t)}{(t^2+1)(\sigma^2(t)+1)}e_{-\frac{1}{2}}(t,0) \qquad (4.52)\\ \|B(t)\| &\leq \gamma(t)=\frac{t\sigma(t)}{(t^3+1)(\sigma^3(t)+1)}\end{aligned}$$

also

$$\begin{aligned}\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}(l(s)+\gamma(s))\triangle s &\leq \int_0^{+\infty}\frac{e_{-\frac{1}{2}}(s,0)}{\left(1-\frac{1}{2}\mu(s)\right)e_{-\frac{1}{2}}(s,0)}\\ &\quad\times\left(\frac{s^2+\sigma^2(s)}{(s^3+1)(\sigma^3(s)+1)}+\frac{s\sigma(s)}{(s^3+1)(\sigma^3(s)+1)}\right)\triangle s\\ &\leq \frac{4}{3}\int_0^{+\infty}\frac{s^2+\sigma^2(s)+s\sigma(s)}{(s^3+1)(\sigma^3(s)+1)}\triangle s\\ &= \frac{4}{3}\int_0^{+\infty}\left(\frac{-1}{s^3+1}\right)^{\triangle}\triangle s \qquad (4.53)\\ &= \frac{4}{3}<\infty\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t} \frac{v(s)}{h(\sigma(s))} \triangle s &\leq \int_{0}^{+\infty} \frac{1}{\left(1-\frac{1}{2}\mu\,(s)\right) e_{-\frac{1}{2}}(s,0)} \frac{2\,(s+\sigma(s))}{(s^2+1)\,(\sigma^2(s)+1)} e_{-\frac{1}{2}}(s,0) \triangle s \\
&\leq \frac{4}{3}\int_{0}^{+\infty} \frac{2\,(s+\sigma(s))}{(s^2+1)\,(\sigma^2(s)+1)} \triangle s \\
&= \frac{4}{3}\int_{0}^{+\infty} \left(\frac{-2}{s^2+1}\right)^{\triangle} \triangle s \\
&= \frac{8}{3} < +\infty \qquad (4.54)
\end{aligned}$$

All statements of Theorem 4.4.4 *are verified. Therefore, we conclude that equation* (4.47) *is globally practically uniformly h-stable*

Conclusion

This work presents a series of results concerning the h- stability of dynamical systems and integral inequalities on time scales.The field of dynamic equations on non-uniform time domains was initiated in 1988 [62] and has received considerable attention in recent years. In particular, the use of integral approximations to study various stability properties remains an active and promising area of research.We aim to investigate qualitative aspects, such as uniform exponential stability, in selected classes of perturbed dynamic systems. This objective leads us to explore specific types of nonlinear integral inequalities within the framework of time scales. These inequalities play a key role in ensuring that stability characteristics of the unperturbed (nominal) system can be extended to the perturbed system, provided certain sufficient conditions are met.

Finnaly, the idea of extending and generalizing some existing Pachpette-Gamidov type on time scales and fractionnal type integral inequalities is carried out.

International Publications :

1. **F. Said , B. Kilani , K. Boukerrioua**,**On Generalized Fractional Integral Inequalities And Applications to global solutions of fractional differential equations. J. Appl. Math. & Informatics, vol. 42(2024**), **No. 4, 915-930.**

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