وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

BADJI MOKHTAR UNIVERSITY- Annaba-

جامعة باجي مختار
- عنابة-

Faculty of Sciences

Department of Mathematics

Year: 2024/2025

THESIS

Presented to Obtaining the Doctorate Degree

MODELING THE DYNAMICS OF CERTAIN EMERGING EPIDEMICS

Stream

Mathematics

Specialty

Mathematics and Applications

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Faculté des Sciences

Département de Mathématiques

Année :2024/2025

THÈSE

Présenté en vue de l'obtention du diplôme de Doctorat en mathématiques

MODELISATION DE LA DYNAMIQUE DE CERTAINES EPIDEMIES EMERGENTES

Filière
Mathématiques
Spécialité
Mathématiques et Applications
Par
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Acknowledgments

First and foremost, I would like to express my gratitude to Allah above all, for the strength and determination. He has bestowed upon me to complete this work.

I extend my deepest gratitude to my thesis advisor, **M. Kouche Mahieddine**, professor at the University of Annaba and my co-supervisor **M. Ainseba Bedr eddine** professor at the University of Bordeaux, for their assistance, support, guidance, and the trust he placed in me by agreeing to supervise my work.

I also extend my sincere thanks to all the members of the jury for the honor they bestowed upon me by agreeing to examine this thesis. I would like to express my gratitude **M. Chorfi Lahcene** professor at the University of Annaba, he granted me by agreeing to chair this thesis committee.
Also **M. Djebabla Abdelhak**, professor at the University of Annaba, **M. Sahari Mohamed Lamine**, M.C.A at the University of Annaba, **M. Maouni Messaoud**, professor at the University of 20 august 1955 Skikda, for the honor they bestowed upon me by participating in this committee as examiners, and for the time devoted to reading this thesis.

I extend my sincere thanks to my parents, my Husband, and all my family for their continuous companionship, assistance, support, and encouragement during the entire process of completing this thesis.
I extend my sincere thanks to my friends from all over the world, especially to Zineb Bouslah who supported me spiritually and morally during this journey.

Finally, I would like to thank everyone who has contributed, directly or indirectly, to the completion of this thesis.

Résumé :

La présente étude discute deux modèles mathématiques généraux non-linéaires.

Le premier travail développe un modèle mathématique fractionnaire avec fonction d'incidence général et un retard d'application au COVID-19 en Algérie, suite à la maladie provoquée par l'épidémie du nouveau coronavirus apparue en Chine au mois de décembre 2019. Analytiquement, les propriétés fondamentales de la solution sont établies et évoquées. En utilisant la théorie des dérivées d'ordre fractionnaire, la stabilité de l'équilibre a été analysée. Pour étayer les résultats analytiques, des simulations numériques sont effectuées pour identifier les facteurs qui affectent de manière significative la capacité de la maladie à se propager. Le logiciel Matlab a été utilisé pour les simulations numériques.

Dans le deuxième travail, pour explorer le comportement de la solution où la fonction d'incidence est plus générale, un modèle épidémique fractionnaire SEIR sensible, exposé, infecté est présenté avec un taux d'incidence général non linéaire a été présenté, où la dérivée est le sens de Caputo. Après avoir prouvé les propriétés fondamentales de la solution telles que l'existence et la positivité, nous utilisons l'approche matricielle de nouvelle génération pour obtenir la valeur du numéro de reproduction de base noté $\mathscr{R}_0$. Nous démontrerons que si $\mathscr{R}_0$ est inférieur à un, alors il existe un équilibre unique sans maladie qui est à la fois localement asymptotiquement stable en utilisant les outils théoriques du calcul fractionnaire, lorsque $\mathscr{R}_0>1$ l'équilibre endémique est localement asymptotiquement stable. De plus, en utilisant une fonction de Lyapunov adaptée, nous prouverons la stabilité globale de l'équilibre sain et établirons des exigences suffisantes pour les deux points d'équilibre. Enfin, nous fournissons quelques simulations numériques pour démontrer nos principales conclusions.

Mots clés : L'ordre fractionnaire, le modèle épidémique, l'analyse de stabilité, le retard, les simulations numériques.

Abstract

In this study, two general nonlinear mathematical models was discussed.

The first work develops a fractional mathematical model with general incidence rate and time delay in application to COVID-19 in Algeria owing to the disease caused by new coronavirus pandemic that emerged in China in December 2019. Analytically, the well-posedness of this model is established and discussed. Using the theory of fractional order derivative, the equilibrium stability was analyzed. In order to support analytic results, numerical simulations were carried out to identify the factors that significantly affect the disease's ability to spread Matlab software was used for the numerical simulations.

In the second work, to explore the behavior of the solution where the incidence function is more general, a fractional Susceptible, Exposed, Infected and Recovered $SEIR$ epidemic model has been presented, where the derivative is the sense of Caputo. After proving the basic proprieties of the solution, we use the next generation matrix approach to get the value of the fundamentalreproduction number noted $\mathscr{R}_0$. We will demonstrate that if $\mathscr{R}_0$ is smaller than one, then there exists a unique disease-free equilibrium that is both locally asymptotically stable by using the theory tools of fractional calculus, but when $\mathscr{R}_0 > 1$ the endemic equilibrium is locally asymptotically stable. Furthermore, using a suitable of Lyapunov function, we will prove the global stability of the healthy equilibrium and establish sufficient requirements for both equilibrium point. Finally, we provide some numerical simulations to demonstrate our main findings.

Keywords: Fractional order derivative, Epidemic model, Stability analysis, the time delay, Numerical simulations.

ملخص :

في هذه الدراسة تمت مناقشة نموذجين رياضيين نظريين عامين غير خطيين .

يوضح العمل الأول نموذجًا رياضيًا كسريًا مع معدل الإصابة العام والتأخير الزمني مع التطبيق على كوفيد-19 في الجزائر وهو المرض الناجم عن وباء فيروس كورونا الجديد الذي ظهر في الصين في ديسمبر 2019. ومن الناحية التحليلية، تم اثبات حسن وضع هذا النموذج. باستخدام نظرية مشتقات الرتبة الكسرية، تم تحليل استقرار التوازن. ولدعم النتائج التحليلية يتم إجراء عمليات محاكاة عددية لتحديد العوامل التي تؤثر بشكل كبير على قدرة المرض على الانتشار وذلك باستخدام برنامج ماتلاب.

في العمل الثاني، لاستكشاف سلوك الحل حيث تكون دالة الإصابة أكثر عمومية، تم تقديم نموذج وبائي كسري مع معدل حدوث غير خطي عام، حيث يكون المشتق هو معنى كابوتو. بعد إثبات الخصائص الأساسية للحل مثل الإيجابية والحدود، قمنا باستخدام طريقة مصفوفة الجيل التالي للحصول على قيمة الأساسية رقم التكاثر الأساسي.

سنثبت أنه باستخدام الأدوات النظرية لحساب التفاضل والتكامل الكسري إذا كان رقم *التكاثر الأساسي* أصغر من الواحد، فهذا يعني وجود توازن فريد خالٍ من الأمراض ومستقر محليًا، ولكن عندما يكون *رقم* التكاثر الأساسي اكبر من الواحد كل فرد سينقل العدوى لأكثر من شخص آخر في المتوسط. علاوة على ذلك، باستخدام دالة ليابونوف المناسبة، سنثبت الاستقرار العالمي للتوازن الصحي ونضع شروط كافية لكلتا نقطي التوازن. وأخيرًا، نقدم بعض عمليات المحاكاة الرقمية لتوضيح النتائج الرئيسية التي .توصلنا إليها ذلك باستخدام برنامج ماتلاب.

الكلمات المفتاحية : المشتقة الكسرية، النموذج الوبائي، تحليل الاستقرار، التأخير الزمني، المحاكاة العددية.

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Notations

1. Let $\Omega = [a,b](-\infty \leq a < b \leq \infty)$ and $1 \leq p \leq \infty$. For $1 \leq p < \infty$, $L^p(\Omega)$ is the set of measurable function $f : \Omega \longrightarrow \mathbb{R}$ where $\int_\Omega |f(x)|^p dx < \infty$.

2. For $p = \infty$, $L^\infty(\Omega)$ is the set of measurable function such that f is bounded every where in Ω.

3. Γ gamma function.

4. E_α Mittag-Leffler function with one parameters.

5. $E_{\alpha,\beta}$ Mittag-Leffler function with two parameters.

6. I_a^α fractional integral of f of order α.

7. D^α fractional derivative in the Caputo sense.

8. $\mathbb{N}$ the set of positive integer numbers.

9. $\mathbb{R}$ the set of real numbers.

10. $\mathbb{R}_+$ The set of positive reals numbers.

11. $\mathcal{R}e(z)$ real part of number z.

12. $\mathcal{C}(A;B)$ the set of continuous function from A to B.

13. $|x| = \sum_{i=1}^{n} x_i$ (norm in $\mathbb{R}^n$) .

Let $\|x\| = \left(\sum_{k=1}^{2} x_k^2\right)^{\frac{1}{2}}$ the norm of $x = (x_1, x_2)^T \in \mathbb{R}^2$. Let Ω be an open and bounded set in $\mathbb{R}$ with smooth boundary $\partial\Omega$ and mes $\Omega > 0$; $0 = (0,0) \in \mathbb{R}^2$, $E = C(\bar{\Omega};\mathbb{R}), \bar{\Omega} = \Omega \cup \partial\Omega$. Let $(E, \|.\|)$ be a Banach space. denoted by $C_\tau = C([-\tau, 0]; E) = \{v : [-\tau, 0] \longrightarrow E\}$ the Banach space equipped with the norm $\|v\|_{C_\tau} = \max\{\|v\| : t \in [-\tau, 0]\}$, C_τ^+ be the positive cone of C_τ.

General Introduction

The study of infectious disease transmission is an aspect of epidemiology in an individual and the features leading to their incidence respectively. Numerous writers employed mathematical models to identify the most successful control strategies. [16, 17, 40, 42, 43, 45, 50].

Since time delays are utilized to explicate observed oscillations and represent biological realities like immunity and the disease's latent period, they are frequently and significantly used in epidemic disease modeling. As a case study, the Time delays used to proved the intervals throughout which immune-compromised individuals recover [9, 17, 18, 23, 28, 33, 44, 47, 48, 51].

It is worth noting that the majority of previous works have concentrated on epidemic models of integer-order derivatives type. Recently, it was proved [20], using the fractional order derivative, the initial values have the same form as that for integer order differential equations which is more applicable for mathematical modelling. Furthermore, fractional derivative include the memory and genetic effects that play a crucial part in the spread of infectious diseases and makes the situation more plausible. In [41] the authors proposed a fractional mathematical model of typhoid fever disease. The modelling of epidemic models using fractional order derivative is discussed for example in [8, 23, 25, 31, 35, 37, 50].

The delay to fractional-order disease's incubation period that should be considered in the epidemic model. Naresh et al. [33] examined how a dynamically behaving of a delayed fractional order epidemic model with bilinear functional response. Kumar & Erturk [23] proposed a fractional mathematical model with delay to study the COVID-19 epidemic in China. They used the fractional operator Liouville-Caputo approach. Danane et al. [7]created a mathematical modelfor describing of COVID-19. Different other types of fractional-order epidemic systems with delays was considered in [1, 22, 40, 47, 45, 50].

Following these works, we investigate the qualitative behaviour of a class of fractional SEIR epidemic models with a more general incidence rate function and time delay to incorporate latent infected individuals. We first prove positivity and boundedness of solutions of the system. The basic reproduction number $\mathcal{R}_0$ of the model is computed using the method of next generation matrix and we prove that if $\mathcal{R}_0 < 1$ the healthy equilibrium is locally asymptotically stable and when $\mathcal{R}_0 > 1$ the system admits an unique endemic equilibrium which is locally asymptotically stable. Moreover, using a suitable Lyapunov function and some results about the theory of stability of differential equations of delayed fractional order type, we give a complete study of global stability for both healthy and endemic steady states. The model is used to describe the COVID-19 outbreak in Algeria at its beginning in February 2020. A numerical scheme [3], based on Adams-Bashforth-Moulton method, is used to run the numerical simulations and show that the number of new infected individuals will peak around late July 2020. Further, numerical simulations show that around 90% of population in Algeria will be infected. Compared with WHO data our results are much more close to real data. Our model with fractional derivative and delay can then better fit the data of Algeria at the beginning of infection and before the lock and isolation measures [38].

In chapter 3, the delayed fractional epidemic $SEIR$ model with diffusion and general incidence function has been presented. First, we have proving the non-negativity and boundlessness of solution using the tools of fractional calculus. Next, the basic reproduction number is determined using the approach of next-generation matrix. If

$\mathscr{R}_0 < 1$ we will demonstrate the local stability of free-steady states for $\tau \in [0, \tau^*)$ and will investigate of existence of Hopf bifurcation when $\tau = \tau^*$. For $\mathscr{R}_0 > 1$ we will prove the local stability of the endemic equilibrium point with and without time delay τ and for critical value of $\tau = \tau^*$. Finally, Using the Lyapunov direct method we will prove the global stability of the free-steady states and we will give some sufficient conditions to establish the global stability of endemic equilibrium.

CHAPTER

1 PRELIMINARIES

1.1 Recall of differential equations ordinary

Definition 1.1.1 *Let* $U \subseteq \mathbb{R}, V \subseteq \mathbb{R}^n$ *and* $\Upsilon \subseteq \mathbb{R}^k$. *Suppose that* $f : U \times V \times \Upsilon \longrightarrow \mathbb{R}^n$ *is a smooth function. An ordinary differential equation is an equation of the form*

$$\dot{x} = f(t, x, \lambda) \tag{1.1}$$

where the dot denotes differentiation with respect to the independent variable t, *the dependent variable* x *is a vector of state variables and* λ *is a vector of parameters.*

The highest derivative appearing in f is called the order of differential equation.
Types of differential equations :
Differential equations can be classified : autonomous deferential equation when the function x does not depend explicitly on the independent variable. But, when the function f does depend explicitly on t, then the corresponding differential equation is called non-autonomous
The system of ordinary differential equations is given by

$$\begin{cases} x_1^{(k)} &= f_1(t, x, x^{(1)}, x^{(2)}, ..., x^{(k-1)}), \\ \vdots &= \vdots \\ x_n^{(k)} &= f_n(t, x, x^{(1)}, x^{(2)}, ..., x^{(k-1)}), \end{cases}$$

A matrix all whose eigenvalues satisfy $Re(\alpha_j) < 0$ is also known as a Hurwitz matrix. The Routh-Hurwitz criterion states that a real matrix is Hurwitz if and only if the following determinants are strictly positive

$$\det\begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{2k-1} & a_{2k-2} & a_{2k-3} & a_{2k-4} & a_{2k-5} & a_{2k-6} & \cdots & a_k \end{pmatrix} > 0.$$

for $1 \leq k \leq n$. Here the numbers a_j are the coefficients of the characteristic polynomial of A

$$det(xI - A) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n.$$

and $a_j > 0$ for $j \geq n$.

Definition 1.1.2 *A fixed point x_0 of $f(x)$ is called stable if for any given neighborhood $U(x_0)$ there exists another $V(x_0) \subseteq U(x_0)$ such that any solution starting in $V(x_0)$ remains in $U(x_0)$ for all $t \geq 0$. A fixed point which is not stable will be called unstable.*

Definition 1.1.3 *The gamma function $\Gamma(x)$ is defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1.2}$$

Definition 1.1.4 *The Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+$ is defined by*

$$(\mathcal{L} f)(s) = \int_0^\infty e^{-st} f(t) dt \quad (s \in \mathbb{C} \tag{1.3}$$

1.2 Stability of Lyapunov direct method

let us consider the autonomous differential equation

$$\dot{x} = f(x) \tag{1.4}$$

defined on an open set $U \subset \mathbb{R}^n$ and its flow φ_t.

Definition 1.2.1 *(Stability) We state that the equilibrium point x_0 of the differential equation (1.4) is stable, if for each $\epsilon \geq 0$, there is a number $\delta > 0$ such that*

$$|\varphi_t(x) - x_0| < \epsilon \text{ for all } t \geq 0 \text{ whenever } |x - x_0| < \delta.$$

Definition 1.2.2 *(Asymptotically stable) A solution $t \rightarrow \varphi_t(x_0)$ of the differential equation (1.4) is asymptotically stable if it is stable and there is a constant $\epsilon > 0$ such that*

$$\lim_{t \longrightarrow \infty} |\varphi_t(x) - \varphi_t(x_0)| = 0 \text{ whenever } |x - x_0| < \epsilon.$$

Definition 1.2.3 *A solution that is not stable is called unstable.*

Definition 1.2.4 *Let be $x, x_0 \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}$ is an open set with $x_0 \in U$.*
The function $V : U \longrightarrow \mathbb{R}$ is called a Lyapunov function for the differential equation (1.4) at x_0 provided that

i) $V(x_0) = 0$.

ii) $V(x) > 0$ *for* $x \in U - \{x_0\}$.

iii) the function $x \longrightarrow grad V(x)$ is continuous for $x \in U - \{x_0\}$ and on this set $\dot{V}(x) = grad V(x).f(x) \leq 0$.

in addition if,

iv) $\dot{V}(x) < 0$ *for* $x \in U - \{x_0\}$.

the V is called a strict Lyapunov function.

1.2.1 Ordinary differential equations with delay

Suppose $\tau \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$ the set of the real number, $\mathbb{R}^n$ is an dimensional linear vector space over the reals with the norm $|.|$. $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval into with the topology of uniform convergence $|.| = \sup\limits_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$.
The retarded functional differential (denotes RFDE) equation is given by

$$\dot{x}(t) = f(t, x(t), x_t), \tag{1.5}$$

where $\tau > 0$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous. $x_t \in \mathcal{C}$ defined for $\theta \in [-\tau; 0], t \in [s; s + A], A \geq 0, s \in \mathbb{R}$ by

$$x_t = x(t + \theta), \tag{1.6}$$

For a $\varphi : [-s; 0] \longrightarrow \mathbb{R}$ one would certainly want φ to satisfy enough smoothness conditions to ensure that finding a solution of equation (1.4) for $t \geq s$ satisfying $x(s + \theta) = \varphi(\theta), -s \leq \theta \leq 0$, would be equivalent to finding a solution of the integral

$$\begin{aligned} x(t) = \varphi(0) + \int_s^t f(\xi, x(\xi), x(\xi - \tau))d\xi, \ \ t \geq s \\ x(s + \theta) = \varphi(\theta), \ \ -\tau \leq \theta \leq 0. \end{aligned} \tag{1.7}$$

Theorem 1.2.1 *[13] Suppose Ω is an open set in in $\mathbb{R} \times \mathcal{C}$, $f : \Omega \longrightarrow \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitizian in φ in each compact set in Ω. If $(s, \varphi) \in \Omega$, then there is a unique solution of equation through (s, φ).*

Definition 1.2.5 *[13] We say that $V : \mathcal{C} \longrightarrow \mathbb{R}$ is a Lyapunov on a set G in C relative to equation (1.4), if V continuous on $\bar{G}$, the closure of G, and $\dot{V} \leq 0$ on G. Let*

$$\begin{aligned} S = \{\varphi \in \bar{G}(\varphi) = 0\}, \\ M = Largest\ set\ in\ S\ that\ is\ invariant\ with\ respect\ to\ equation\ (1.4). \end{aligned} \tag{1.8}$$

Theorem 1.2.2 *[13] Suppose $f : \Omega \longrightarrow \mathbb{R}^n$ takes $\mathbb{R}\times$(bounded sets of $\mathcal{C}$) into bounded sets of $\mathbb{R}^n$. $U, V, W : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous non decreasing functions, $u(\xi)$ and $v(\xi)$ are positive for $\xi > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times \mathcal{C} \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned} U(|\varphi(0)|) \leq V(t, \varphi) \leq U(|\varphi|), \\ V^{\cdot}(t, \varphi) \leq -W(|\varphi(0)|, \end{aligned} \tag{1.9}$$

then the solution $x = 0$ of equation (1.4) is uniformly stable. If $u(\xi) \longrightarrow \infty$ as $s \longrightarrow \infty$, the solution of equation (1.4) are uniformly bounded . If $W(\xi) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

1.2.2 Fractional differential equations

In this subsection, we recall some definitions and lemmas about fractional calculus, which will be used in deriving the main results where the derivative is the sense of the Caputo.

Definition 1.2.6 *[36] The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in \mathcal{C}^n(^+, \mathbb{R})$ is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is a positive integer such that $\alpha \in (n-1, 1]$. Also, the corresponding fractional integral of ordre α with $Re(\alpha) > 0$ is given by

$$I^\alpha_{[0,t]} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(.)$ is the Gamma function.

Proposition 1.2.1 *[11] Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be such that $D^\alpha f(t)$ and $D^\alpha g(t)$ exist almost everywhere and let $a_1, a_2 \in \mathbb{R}$. Then $D^\alpha(a_1 f(t) + a_2 g(t))$ exists almost everywhere, and*

$$D^\alpha(a_1 f(t) + a_2 g(t)) = a_1\, D^\alpha f(t) + a_2\, D^\alpha g(t).$$

Further, the Caputo fractional derivative for a constant function is zero.

Lemma 1.2.1 *[34] Suppose that $f \in \mathcal{C}[a,b]$ and $D^\alpha f \in \mathcal{C}[a,b]$ with $0 < \alpha \leq 1$. Then there exists $\xi(x) \in [a,x]$, such that*

$$f(x) = f(a) + \tfrac{1}{\alpha} D^\alpha f(\xi)(x-a)^\alpha.$$

Based on the previous Lemma we have the following corollary.

Corollary 1.2.1 *Suppose that $f \in \mathcal{C}([a,b])$ and $D^\alpha f(t) \in \mathcal{C}([a,b])$.*
For $0 \leq \alpha \leq 1$, if $D^\alpha f(t) \geq 0, (resp, D^\alpha f \leqslant 0)$ $\forall t \in (a,b)$, then $f(t)$ is nondecreasing (resp, nonincreasing) for each $t \in [a,b]$.

Definition 1.2.7 *The constant point x^* is an equilibrium point of the Caputo-fractional model*

$$D^{\alpha}x(t) = f(t, x(t)),$$

if and only if $f(t, x^*) = 0$ for all $t > 0$.

Lemma 1.2.2 *[25] Let $\alpha \in (0,1)$ and consider a continuous function $x : [t_0, \infty) \longrightarrow \mathbb{R}$ satisfying the following condition*

$$D^{\alpha}x(t) + \mu x(t) \leq \nu, \quad t_0 \geq 0, \quad \mu, \nu \in \mathbb{R}, \ \mu \neq 0.$$

Then, we have the inequality

$$x(t) \leq (x_0 - \frac{\nu}{\mu})E_{\alpha}[-\mu(t-t_0)^{\alpha}] + \frac{\nu}{\mu},$$

for all $t \geq t_0$, where E_{α} is the Mittag-Leffler function of one parameter defined by

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}.$$

We can now state the following existence result for fractional order equations.

Lemma 1.2.3 *Let $f : [t_0, \infty) \times \Omega \longrightarrow \mathbb{R}^n$ be a function satisfying the Lipschitz condition on x and consider the following fractional order equation*

$$D^{\alpha}x(t) = f(t, x(t)), \ t > 0, \quad \alpha \in (0,1],$$

with the initial condition $x(t_0) = x_0$. Then the above system has a unique solution.

Lemma 1.2.4 *[8] Let $x^* \in \Omega \subset \mathbb{R}^n$ be an equilibrium point of the system*

$$D^{\alpha}x(t) = f(t, x(t)), \quad t \geq t_0,$$

and let $V(t,x) : [t_0, \infty) \times \Omega \longrightarrow \mathbb{R}$ *be a continuously differentiable function such that*

$$W_1(x) \leq V(t,x) \leq W_2(x),$$
$$D^\alpha V(t,x) \leq -W_3(x),$$

for $t \geq t_0$ *and* $x \in \Omega$, *where* $W_i(x)$, $i = 1,2,3$ *are continuous and positively defined functions on* Ω. *Then* x^* *is uniformly asymptotically stable.*

Proposition 1.2.2 *[2] Consider the following fractional system*

$$\begin{cases} D^\alpha x(t) = f(x,y), \\ D^\alpha y(t) = g(x,y),\ 0 \leq \alpha < 1, \end{cases} \tag{1.10}$$

where the fractional derivative is the sense of Caputo. The equilibrium point $X^* = (x^*, y^*)^T$ *of system (1.10) is the solution of*

$$f(x^*, y^*) = g(x^*, y^*) = 0. \tag{1.11}$$

X^* *is locally asymptotically stable if all the eigenvalues of Jacobian matrix satisfies*

$$|arg(\lambda)| > \tfrac{\alpha\pi}{2}. \tag{1.12}$$

Put $P(\lambda)$ *the characteristic polynomial of linearized system of (1.10)*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n, \tag{1.13}$$

for $n = 2$, *the conditions for (1.12) are either Routh-Hurwitz conditions if*

$$a_1 < 0, 4a_2 > (a_1)^2, \left|\tan^{-1}\left(\frac{\sqrt{4a_2 - (a_1)^2}}{a_1}\right)\right| > \frac{\alpha\pi}{2}.$$

Let f be a continuous function on $L^1([a,b],\mathbb{R})$, a fractional integral of order in the Riemann-Liouville sense corresponding to, a fractional integral of order $\alpha \in (0,1)$ in the Riemann-Liouville sense corresponding to t defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{s-1} f(s) ds, \tag{1.14}$$

where integral on right side is point wise defined on $(0, \infty)$.

Lemma 1.2.5 *[5] Let $x \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ be a continuous and differential function. Then, for any $t \geq 0$ and $0 < \alpha \leq 1$*

$$D^{\alpha}\left(x(t) - x^* - x^* ln\frac{x(t)}{x^*}\right) \leq \left(1 - \frac{x^*}{x(t)}\right) D^{\alpha} x(t).$$

Theorem 1.2.3 *[31] The equilibrium solutions x^* of the autonomous system*

$$D^{\alpha} x(t) = f(x(t)), x(t_0) = x_0, \tag{1.15}$$

is locally asymptotically stable if all the eigenvalues λ_i of the Jacobian matrix evaluated at the equilibrium points satisfy

$$|arg(\lambda_i)| > \frac{\alpha\pi}{2}, 0 < \alpha < 1$$

Definition 1.2.8 *[2] The discriminate $D(f)$ of a polynomial*

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n, \tag{1.16}$$

is defined by

$$D(f) = (-1)^{\frac{n(n-1)}{2}} R(P, P'), \tag{1.17}$$

where P' is the derivative of P. If $g(x) = x^n + a_1 x^{l-1} + a_2 x^{l-2} + \ldots + b_l$, then $R(f, g)$ is the determinant of the corresponding Sylvester $(n+l) \otimes (n+l)$ matrix.
The determinant is very important to define the nature of the roots of $f(x) = 0$. For $n = 3$, the characteristic equation takes the form

$$P(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3, \tag{1.18}$$

then,

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2. \quad (1.19)$$

Theorem 1.2.4 *Let x^* be a equilibrium point of system (1.9) and (1.17) his corresponding characteristic polynomial, then*

$$| arg\lambda | > \frac{\alpha\pi}{2} \quad (1.20)$$

if one the following cases satisfies :

1 *If $D(P) > 0$, $a_1 > 0$, $a_3 > 0$, $a_1a_2 - a_3 > 0$ (necessary and sufficient condition).*

2 *If $D(P) < 0$, $a_1 \geq 0$, $a_2 \geq 0$, $a_3 > 0$, (the condition (1.19) satisfied for $\alpha < \frac{2}{3}$).*

3 *If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 = a_3$ (the condition (1.19) satisfied for all $\alpha \in [0,1)$).*

Lemma 1.2.6 *[4] Let $x(t) \in \mathbb{R}^+$ be a continuous and differential function. Then, for any $t \geq t_0, 0 \leq \alpha \leq 1$, and $\bar{x} > 0$, we have*

$$D^\alpha(\psi(x(t)) \leq (1 - \frac{g(\bar{x})}{g(x(t))})D^\alpha x(t), \quad (1.21)$$

where

$$\psi(x(t)) = x - \bar{x} - \int_{\bar{x}}^{x(t)} \frac{g(\bar{x})}{g(s)} ds, \quad (1.22)$$

with $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a differential and increasing function.

1.2.3 Fractional ordinary differential equations with delay

In this section, we outline a list of important notations, definitions, and lemmas that will be used in our main results.

Consider the following initial value problem (1.22) for fractional delay differential equation :

$$\begin{cases} D^\alpha x(t) &= f(t, x(t), x(t-\tau)),\ t \geq 0,\ \tau > 0,\ n-1 < \alpha \leq n \\ x(t) &= \varphi(t),\ -\tau \leq t \leq 0. \end{cases} \quad (1.23)$$

where D^α, denotes the Caputo derivative of order α. We can now state the following existence result for fractional differential equations.

Theorem 1.2.5 *[6] Let $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be a continuous function and satisfies the following Lipschitz condition with respect to the second variable: there exists a non-negative continuous function $g:[0,T]\times\mathbb{R}^n\to\mathbb{R}_+$ such that*

$$\|f(t,x_1,y)-f(t,x_2,y)\|\leq g(t,y)\|x_1-x_2\|$$

for all $t\in[0,T], x_1,x_2,y\in\mathbb{R}^n$. Then, the following delayed fractional problem

$$D^{\alpha}x(t)=f(t,x(t),x(t-\tau)),\ t\in[0,T],\quad \alpha\in(0,1]$$

with the initial condition

$$x(t)=\varphi(t),\ t\in[-\tau,0],$$

has a unique global solution $u(.,\varphi)$ on the interval $[-\tau,T]$.

Lemma 1.2.7 *[9] Let $\tau>0$, $\alpha\in(0,1]$, A,B two $(n\times n)$ square matrices and $\varphi\in\mathcal{C}([-\tau,0];\mathbb{R}^n)$. Consider the linear fractional delayed differential system with the Caputo derivative*

$$\begin{cases} D^{\alpha}x(t)=Ax(t)+Bx(t-\tau), \quad t>0, \\ x(t)=\varphi(t),\ t\in[-\tau,0]. \end{cases} \tag{1.24}$$

We define the characteristic equation of system (1.23) by

$$\Delta(s)=\det(s^{\alpha}I_n-A-Be^{-s\tau})=0.$$

If all the roots of the characteristic equation $\Delta(s)=0$ have negative real parts, then the zero solution of system (1.23) is locally asymptotically stable.

Lemma 1.2.8 *[9]*

1. *If all the eigenvalues λ of the matrix $M=A+B$ satisfy $|\arg(\lambda)|>\alpha\frac{\pi}{2}$ and the characteristic equation $\Delta(s)=0$ has a no purely imaginary roots for $\tau>0$, then the zero solution of system (1.23) is locally asymptotically stable.*

2. *Suppose $\tau=0$. If all the eigenvalues λ of M satisfy $|\arg(\lambda)|>\frac{\alpha\pi}{2}$, then the zero solution of (1.23) is locally asymptotically stable.*

Theorem 1.2.6 *The Laplace transform of the Caputo-type fractional derivative of function* $f(t) \in \mathcal{C}^n([t_0, \infty), \mathbb{R})$ *is*

$$\mathscr{L}\{D^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(t_0), \ n-1 < \alpha < n-1.$$

1.3 The basic reproduction number

A crucial idea in epidemiology is the fundamental reproduction number, also called the basic reproduction number denotes $\mathcal{R}_0$ was first created for the study of demography by Diekmann and Heesterbeek in[14] and Van den Driessche and Watmough. It is now widely employed in the research of infectious diseases, and more recently in models of in-host population dynamics, it used to gauge the risk of an epidemic or pandemic in emerging infectious disease. Such that if $\mathcal{R}_0 < 1$, then on average an infected individual produces less than one new infected individual over the course of its infections period, witch implies that the disease equilibrium is locally asymptotically stable and the epidemic can not invade the population, but when $\mathcal{R}_0 > 1$ each infected individual produces, on average more than one new infection i-e the healthy equilibrium is unstable and the disease can invade the population [43]. As a general definition of $\mathcal{R}_0$ " is the expected number of secondary individuals produced in a completely susceptible population, by a typical infective indivudial " [43], he depends by several factors :

1. on the duration of the infectious period.
2. the probability of infecting a susceptible individual during one contact.
3. the number of new susceptible individuals contacted per unit of time.

This threshold behavior is the most important and useful aspect of the $\mathcal{R}_0$ concept. In an endemic infection, we can determine which control measures, and at what magnitude, would be most effective in reducing $\mathcal{R}_0$ below one, providing important guidance for public health initiatives. The magnitude of $\mathcal{R}_0$ is also used to gauge the risk of an epidemic or pandemic in emerging infectious disease. For example, the estimation of $\mathcal{R}_0$ was of critical importance in understanding the outbreak and potential danger from severe acute respiratory syndrome.

Derivations of $\mathcal{R}_0$ from a deterministic model

1. Survival function

Consider a large population and let $F(a)$ be the probability that a newly infected individual remains infectious for at least time a. This is called the survival probability. Also, let $b(a)$ denote the average number of newly infected individuals that an infectious individual will produce per unit time when infected for total time a. Then, $\mathcal{R}_0$ is given by:

$$\mathcal{R}_0 = \int_0^\infty b(a)F(a)da, \tag{1.25}$$

2. Next generation method

we use this method when the population is divided into discrete, disjoint classes.
In the next generation method, $\mathcal{R}_0$ is defined as the spectral radius of the next generation operator. The formation of the operator involves determining two compartments, infected and non-infected, from the model.

Let use assume that there are n compartments of which m are infected. We defined the vector $\bar{x} = x_i, i = 1,...,n,$ where x_i denotes the number or proportion of individuals in the i th compartments. Let $F_i(\bar{x})$ be the rate of appearance of new infections in compartments i and

$$V_i(\bar{x}) = V_i^-(\bar{x}) - V_i^+(\bar{x}),$$

where
$V_i^+(\bar{x})$: is the rate of the transfer of individuals into compartment i by all other means.

$V_i^-(\bar{x})$: is the rate of the transfer of individuals out of the i_{-th} compartments.

The differences $V_i^-(\bar{x}) - V_i^+(\bar{x})$, gives the rate of change x_j, it is assumed that each function is continuously differential at least twice in each variable. Note that F_i should include only infection that are newly arising, but does not include terms which describe the transfer of infectious individuals from one infected to another. Assuming the F_i and V_i meet the conditions outlined in [10] [43] , we can form the next generation matrix (operator) FV^{-1} from matrices of partial derivative of F_i and V_i. Specifically,

$$F = [\frac{\partial F_i(x_0)}{\partial x_j}] \quad V = [\frac{\partial V_i(x_0)}{\partial x_j}], \tag{1.26}$$

where $i, j = 1, ..., m$ and x_0 is the disease-free equilibrium. The entries of FV^{-1} give the rate at which infected individuals in x_j produce new infections in x_i, times the average length of time and individual spends in a single visit to compartment j, $\mathcal{R}_0$ is given by the spectral radius (dominant eigenvalue) of the matrix FV^{-1}.

CHAPTER 2

STABILITY OF A FRACTIONAL-ORDER SEIR EPIDEMIC MODEL WITH TIME DELAY AND GENERAL INCIDENCE RATE

2.1 The model

Denote by $N(t)$ the total population size at the time t. We assume that $N(t)$ is divided into four compartments which are : susceptible individuals $S(t)$, exposed individuals $E(t)$, infected individuals $I(t)$ and recovered individuals $R(t)$ at a time t. The susceptible class S consists of individuals who are at risk of catching infection due to close contact with infected individuals. The exposed class E are revealed individuals, but not yet infectious. The infected class I consists of individuals who have already caught the disease, and they can transfer it to susceptible. The recovered class R consists of people who were infected and are now well. Indicate by λ the proportion of vulnerable people recruited, μ the overall death rate, η the mortality of infected people and γ the rate at move from the infected to the recovered compartment. The spreading behavior of the

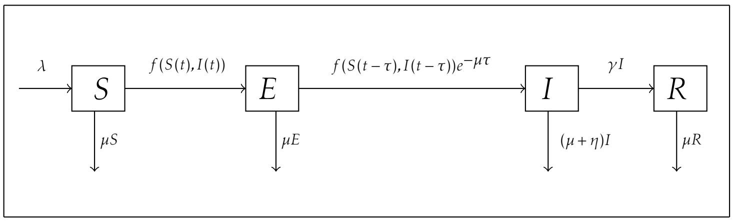


Figure 2.1: The model schematic.

disease is thus guided by the following delayed fractional system :

$$\begin{cases} D^{\alpha}S(t) &=& \lambda - f(S(t),I(t)) - \mu S(t), \\ D^{\alpha}E(t) &=& f(S(t),I(t)) - f(S(t-\tau),I(t-\tau))e^{-\mu\tau} - \mu E(t), \\ D^{\alpha}I(t) &=& f(S(t-\tau),I(t-\tau))e^{-\mu\tau} - (\mu+\eta+\gamma)I(t), \\ D^{\alpha}R(t) &=& \gamma I(t) - \mu R(t), \end{cases} \tag{2.1}$$

as well as D^{α} is the Caputo fractional-order derivative with $0 < \alpha \leq 1$. The following initial conditions are added to system (2.1).

$$S(\theta) = \varphi_1(\theta),\ E(\theta) = \varphi_2(\theta),\ I(\theta) = \varphi_3(\theta),\ R(\theta) = \varphi_4(\theta),\ \ \theta \in [-\tau, 0], \tag{2.2}$$

where $\varphi_i \in \mathcal{C}([-\tau,0];\mathbb{R})$ are non negative such that $\varphi_i(0) > 0$ for $i = 1,2,3,4$. We assume that the incidence function f is always positive, continuous and satisfy for all $S \geq 0, I \geq 0$ the following conditions [16]

$$\begin{array}{ll} (H1) & f(0,I) = f(S,0) = 0, \\ (H2) & \frac{\partial f(S,0)}{\partial S} = 0, \\ (H3) & \frac{\partial f(S,I)}{\partial S} > 0, \\ (H4) & \frac{\partial f(S,I)}{\partial I} > 0, \\ (H5) & \frac{\partial^2 f(S,I)}{\partial^2 I} \leq 0. \end{array} \tag{2.3}$$

The time delay τ in this model symbolizes the period of incubation, and the term $f(S(t-\tau),I(t-\tau))e^{-\mu\tau}$ indicates the people who were exposed at time $t-\tau$ and survive to time t to time t. Denotes by $\mathcal{C} = \mathcal{C}([-\tau,0];\mathbb{R})$ the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}$ equipped with the sup-norm. The non-negative cône of $\mathcal{C}$ is defined as $\mathcal{C}^{+} = \mathcal{C}([-\tau,0],\mathbb{R}_{+})$. The phase space of system (2.1) is then $\mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathcal{C}$. Because the variable has no bearing on the first and third equations E and R. This subsystem is what the model can be simplified to

$$\begin{cases} D^{\alpha}S(t) &=& \lambda - f(S(t),I(t)) - \mu S(t), \\ D^{\alpha}I(t) &=& f(S(t-\tau),I(t-\tau))e^{-\mu\tau} - (\mu+\eta+\gamma)I(t). \end{cases} \tag{2.4}$$

We will demonstrate the presence and validity of the system solutions in the sections that follow.

Lemma 2.1.1 *For the fractional-order system (2.4) with the beginning conditions (2.2), there is one solution. In addition, each resolution of system (2.4)-(2.2) enters a compact attractive set is positive and is bounded.*

Proof 2.1.1 *By the Lemma 1.2.3 system (2.4) with initial conditions (2.2) have a unique solution on some time interval. To demonstrate that $S \geq 0$, we use contradiction. Let us assume that there is a $\varsigma_1 > 0$ such that $S(t) > 0$ for $t \in [0, \varsigma_1)$, $S(\varsigma_1) = 0$ and $S(t) < 0$ for $t \in (\varsigma_1, \varsigma_1 + \epsilon_1]$ with suitably tiny. Starting with the system's initial equation (2.2), it is evident that $D^\alpha S(t)|_{t=\varsigma_1} = \lambda > 0$ and thus by Lemma 1.2.1there's the ξ_1 to the extent that*

$$S(\varsigma_1 + \epsilon_1) = S(\varsigma_1) + \frac{1}{\alpha} D^\alpha S(\xi_1)\epsilon_1^\alpha,$$

where $\varsigma_1 \leq \xi_1 \leq \varsigma_1 + \epsilon_1$. If we select ϵ_1 small enough, we can see $S(\varsigma_1 + \epsilon_1) > 0$ which contradicts the fact that $S(t) < 0$ in $[\varsigma_1, \varsigma_1 + \epsilon_1]$. Thus, we have $S(t) \geq 0$ for $t \geq 0$.
To demonstrate that $I \geq 0$, supposing, by contradiction, that there is $\varsigma_2 > 0$ such that $I(t) > 0$ for $t \in [0, \varsigma_2)$, $I(\varsigma_2) = 0$ and $I(t) < 0$ for $t \in (\varsigma_2, \varsigma_2 + \epsilon_2]$ with ϵ_2 suitably tiny. Based on the system's second equation (2.4), We've got

$$D^\alpha I(t)|_{t=\varsigma_2} = f(S(t-\tau), I(t-\tau))e^{-\mu\tau} \geq 0,$$

using the Lemma 1.2.1, there's the ξ_2 in which

$$I(\varsigma_2 + \epsilon_2) = I(\varsigma_2) + \frac{1}{\alpha} D^\alpha I(\xi_2)\epsilon_2^\alpha,$$

where $\varsigma_2 \leq \varsigma_3 \leq \varsigma_2 + \epsilon_2$, and thus $I(\varsigma_2 + \epsilon_2) > 0$ it is in opposition to the reality that $I(t) < 0$ for $t \in [\varsigma_2, \varsigma_2 + \epsilon_2]$. We now demonstrate that $S, I > 0$. Considering that there$t_1 > 0$ such that $S(t_1)$ is the minimum of S and $S(t_1) = 0$, then

$$D^\alpha S(t_1) = \lambda > 0,$$

then $D^\alpha S$ is non negative in $[t_1 - \varsigma, t_1 + \varsigma[$ for some $\varsigma > 0$, then by Corollary 1.2.1 S is

strictly increasing function and hence $S(t_1-\varsigma) < S(t_1) = 0$ *which is a contradiction. We can use a similar argument to prove that* $I > 0$.
To prove the boundedness of solution let us define

$$N(t) = e^{-\mu\tau}S(t-\tau) + I(t),$$

the fractional derivative of $N(t)$ *is supplied by*

$$\begin{aligned} D^{\alpha}N(t) &\leq e^{-\mu\tau}D^{\alpha}S(t) + D^{\alpha}I(t), \\ &\leq \lambda - \mu N(t), \end{aligned}$$

through Lemma 1.2.2

$$N(t) \leq \Big(N_0 - \tfrac{\lambda}{\mu}\Big)E_{\alpha}(-\mu t^{\alpha}) + \tfrac{\lambda}{\mu},$$

where $N_0 = S(0) + I(0)$. *The last inequality leads to*

$$\limsup_{t\longrightarrow+\infty} N(t) \leq \tfrac{\lambda}{\mu}.$$

Subsequently, the uniformly bounded and global solution of system (2.4).

$$B = \Big\{0 < S + I \leq \tfrac{\lambda}{\mu}\Big\},$$

is a positive attracting set for system (2.4).

2.2 Steady states

To find steady states of system (2.4), We resolve the ensuing system

$$\begin{cases} \lambda - f(S,I) - \mu S = 0, \\ e^{-\mu\tau}f(S,I) - (\mu+\eta+\gamma)I = 0. \end{cases} \tag{2.5}$$

It is evident that $(S^0,0)^T$, with $S^0 = \frac{\lambda}{\mu}$ is always a solution of (2.5). Consequently, the system (2.5) admits a free steady state $E^0 = (S^0,0)$. To determine the system fundamental reproduction number (2.5), we employ the next-generation matrix approach.

[42, 43].

Lemma 2.2.1 *The key reproduction number of system (2.5) adopts the shape*

$$\mathcal{R}_0 = \frac{e^{-\mu\tau}}{\mu+\eta+\gamma}\frac{\partial f}{\partial I}(S^0,0).$$

Proof 2.2.1 *Put $X=(S,I)^T$, following that the system (2.5) may be expressed as follows :*

$$\begin{aligned} D^\alpha X &= F(X) - V(X),\\ V(X) &= V^-(X) - V^+(X), \end{aligned}$$

where

$$F(X) = \begin{pmatrix} 0 \\ e^{-\mu\tau} f(S,I) \end{pmatrix},$$

and

$$V^+(X) = \begin{pmatrix} \lambda \\ 0 \end{pmatrix},\ V_-(X) = \begin{pmatrix} f(S,I)+\mu S \\ (\mu+\eta+\gamma)I \end{pmatrix}.$$

$F(X)$ represents the pace at which new sick people appear in each of the compartments S and I.

$V^+(X)$ the speed at which people are moved into the compartments S and I using every alternative method, and $V^-(X)$ the rate at which people leave the sections S and I.

Let $\mathcal{F}$ and $\mathcal{V}$ be the Jacobian matrices of $F(X), V(X)$ respectively at E^0, then

$$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ e^{-\mu\tau}\frac{\partial f}{\partial S}(S^0,0) & e^{-\mu\tau}\frac{\partial f}{\partial I}(S^0,0) \end{pmatrix},\quad \mathcal{V} = \begin{pmatrix} \mu + \frac{\partial f}{\partial S}(S^0,0) & \frac{\partial f}{\partial I}(S^0,0) \\ 0 & \mu+\eta+\gamma \end{pmatrix}.$$

Following [43], as we define $\mathcal{R}_0$ as the spectral radius of the next generation matrix $\mathcal{F}\mathcal{V}^{-1}$, with $\mathcal{V}$ non-singular. Using (H_2), we get

$$\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{e^{-\mu\tau}}{\mu+\eta+\gamma}\frac{\partial f}{\partial I}(S^0,0).$$

Theorem 2.2.1 *In case $\mathcal{R}_0 > 1$, thus the system (2.4) have a unique endemic steady state $EE^* = (S^*, I^*)$.*

Proof 2.2.2 *Let (S, I) be a solution of (2.4) such that $I \neq 0$. By using the system first and second equations, we have*

$$\lambda - \mu S = f(S, I) = e^{\mu\tau}(\mu + \eta + \gamma)I,$$

which means

$$S = \frac{\lambda - e^{\mu\tau}(\mu+\eta+\gamma)I}{\mu}.$$

It is evident that S exists if and only if $I < \tilde{I} = \frac{\lambda e^{-\mu\tau}}{\mu+\eta+\gamma}$. In the next, we suppose that $0 < I < \tilde{I}$, where I is how the following equation should be solved.

$$f\left(\frac{\lambda - e^{\mu\tau}(\mu+\eta+\gamma)I}{\mu}, I\right) - e^{\mu\tau}(\mu + \eta + \gamma)I = 0. \tag{2.6}$$

If $I = 0$, we obtain the free steady state E^0. For $I \neq 0$, let H be the function described by

$$H(I) = \frac{f\left(\frac{\lambda - e^{\mu\tau}(\mu+\eta+\gamma)I}{\mu}, I\right)}{I} - e^{\mu\tau}(\mu + \eta + \gamma). \tag{2.7}$$

Using the hypotheses (H_3), we conclude that the time derivative of H is negative. According to the definition of $\mathcal{R}_0$, As we have

$$\lim_{I \longrightarrow 0^+} H(I) = e^{\mu\tau}(\mu + \eta + \gamma)(\mathcal{R}_0 - 1),$$

if $\mathcal{R}_0 > 1$, we leads to

$$\lim_{I \longrightarrow 0^+} H(I) > 0.$$

However,

$$\lim_{I \longrightarrow \tilde{I}} H(I) = -e^{\mu\tau}(\mu + \eta + \gamma) < 0.$$

Thus, by the fundamental theorem of algebra, there exists an unique positive root $0 < I^* < \tilde{I}$ of the equation (2.6). Hence, the system (2.4) has a unique endemic steady state $EE^* = (S^*, I^*)$.

2.3 Locally stability of steady states

We examine the local stability of both free and endemic stable states in this section. Indicate with $(\overline{S},\overline{I})$ one of the two steady states E^0 or EE^*. The linearised system (2.4) around $(\overline{S},\overline{I})$ has the following structure

$$\begin{cases} D^\alpha S(t) &= -\left(\frac{\partial f(\overline{S},\overline{I})}{\partial S}+\mu\right)S(t)-\frac{\partial f(\overline{S},\overline{I})}{\partial I}I(t), \\ D^\alpha I(t) &= -(\mu+\gamma+\eta)I(t)+e^{-\mu\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S}S(t-\tau)+e^{-\mu\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}I(t-\tau). \end{cases} \tag{2.8}$$

Applying the Laplace transform to both system sides (2.8), our findings

$$\begin{cases} s^\alpha L[S(t)] &= s^{\alpha-1}S(0)-\left(\frac{\partial f(\overline{S},\overline{I})}{\partial S}+\mu\right)L[S(t)]-\frac{\partial f(\overline{S},\overline{I})}{\partial I}L[I(t)], \\ s^\alpha L[I(t)] &= s^{\alpha-1}I(0)-(\mu+\gamma+\eta)L[I(t)]+e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S}\Big(L[S(t)] \\ & \quad +\int_{-\tau}^{0}e^{-st}\varphi_1(t)dt\Big)+e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}\Big(L[I(t)]+\int_{-\tau}^{0}e^{-st}\varphi_3(t)dt\Big), \end{cases}$$

where $L[I(t)]$, $L[S(t)]$ are the Laplace transform of $S(t)$ and $I(t)$ respectively. The system mentioned above can be expressed as follows:

$$A(s)\begin{pmatrix} L[S(t)] \\ L[I(t)] \end{pmatrix}=\begin{pmatrix} B_1(s) \\ B_2(s) \end{pmatrix},$$

with

$$\begin{cases} B_1(s) &= s^{\alpha-1}S(0), \\ B_2(s) &= s^{\alpha-1}I(0)+e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S}\int_{-\tau}^{0}e^{-st}\varphi_1(t)dt+e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}\int_{-\tau}^{0}e^{-st}\varphi_3(t)dt, \end{cases}$$

and

$$A(s)=\begin{pmatrix} s^\alpha+\frac{\partial f(\overline{S},\overline{I})}{\partial S}+\mu & \frac{\partial f(\overline{S},\overline{I})}{\partial I} \\ -e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S} & s^\alpha-e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}+\mu+\gamma+\eta \end{pmatrix}.$$

The characteristic polynomial $\Delta(s)$ of $A(s)$ is

$$\begin{aligned} \Delta(s) &= s^{2\alpha}+\left[\frac{\partial f(\overline{S},\overline{I})}{\partial S}+2\mu-e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}+\gamma+\eta\right]s^\alpha \\ & \quad +(\mu+\gamma+\eta)\left(\frac{\partial f(\overline{S},\overline{I})}{\partial S}+\mu\right)-\mu e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}. \end{aligned} \tag{2.9}$$

2.3.1 Local stability of free steady state

The local stability of the free steady state is presented in the following.

Proof 2.3.1 *For $\tau = 0$, the characteristic matrix of the linearised system (2.8) examined at E^0 adopts the form*

$$A = \begin{pmatrix} -\frac{\partial f(S^0,0)}{\partial S} - \mu & -\frac{\partial f(S^0,0)}{\partial I} \\ \frac{\partial f(S^0,0)}{\partial S} & \frac{\partial f(S^0,0)}{\partial I} - (\mu + \gamma + \eta) \end{pmatrix},$$

due to $\frac{\partial f(S^0,0)}{\partial S} = 0$, then the characteristic polynomial of A is supplied by

$$\begin{aligned} P(\lambda) &= \lambda^2 + \Big(2\mu - \frac{\partial f(S^0,0)}{\partial I} + \gamma + \eta\Big)\lambda - \mu\frac{\partial f(S^0,0)}{\partial I} + \mu(\mu + \gamma + \eta), \\ &= \lambda^2 + \Big(\mu + (\mu + \gamma + \eta)(1 - \mathcal{R}_0)\Big)\lambda + \mu(\mu + \gamma + \eta)(1 - \mathcal{R}_0). \end{aligned}$$

Since $R_0 < 1$, every coefficient of P are positive and by Routh-Hurwitz theorem all the roots λ of P have negative real parts, which imply that $|\arg(\lambda)| > \frac{\pi}{2} > \alpha\frac{\pi}{2}$. Using Lemma 1.2.8/(2), we conclude that the free steady state is locally asymptotically stable.

Theorem 2.3.1 *Suppose that $\tau > 0$. If the basic reproduction number less then one, then the free steady state E^0 is locally asymptotically stable.*

Proof 2.3.2 *From (2.9) the characteristic equation at E^0 is given by*

$$s^{2\alpha} + \Big(2\mu - e^{-(\mu+s)\tau}\frac{\partial f(S^0,0)}{\partial I} + \gamma + \eta\Big)s^\alpha - \mu e^{-(\mu+s)\tau}\frac{\partial f(S^0,0)}{\partial I} + \mu(\mu + \gamma + \eta) = 0. \quad (2.10)$$

To prove local stability of E^0, we use Lemma 1.2.8/1 assume by contradiction that the equation (2.10) has a pair of imaginary roots $s = \omega e^{i\frac{\pi}{2}}$, $\omega > 0$. After substituting s into equation (2.10), we obtain

$$\omega^{2\alpha}e^{\alpha\pi i} + \Big(2\mu - e^{-\mu\tau}e^{-i\omega\tau}\frac{\partial f(S^0,0)}{\partial I} + \gamma + \eta\Big)\omega^\alpha e^{\frac{\alpha\pi i}{2}} - \mu e^{-\mu\tau}e^{-i\omega\tau}\frac{\partial f(S^0,0)}{\partial I} + \mu(\mu + \gamma + \eta) = 0,$$

separating real and imaginary parts, we have

$$\begin{cases} A_1\cos(\omega\tau)+A_2\sin(\omega\tau)=A_3, \\ A_2\cos(\omega\tau)-A_1\sin(\omega\tau)=A_4, \end{cases} \tag{2.11}$$

where

$$\begin{aligned}
A_1 &= \mu e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}+\omega^{\alpha}e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\cos(\tfrac{\alpha\pi}{2}),\\
A_2 &= \omega^{\alpha}e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\sin(\tfrac{\alpha\pi}{2}),\\
A_3 &= \omega^{2\alpha}\cos(\pi\alpha)+\omega^{\alpha}\Big(2\mu+\gamma+\eta\Big)\cos(\tfrac{\alpha\pi}{2})+\mu(\mu+\gamma+\eta),\\
A_4 &= \omega^{2\alpha}\sin(\pi\alpha)+\omega^{\alpha}(2\mu+\gamma+\eta)\sin(\tfrac{\alpha\pi}{2}).
\end{aligned}$$

Eliminating τ by squaring and adding the two equations in system (2.11), we obtain

$$\omega^{4\alpha}+A_5\omega^{3\alpha}+A_6\omega^{2\alpha}+A_7\omega^{\alpha}+A_8=0, \tag{2.12}$$

where

$$\begin{aligned}
A_5 &= 2(2\mu+\gamma+\eta)\sin(\pi\alpha)\sin(\tfrac{\alpha\pi}{2})+2(2\mu+\gamma+\eta)\cos(\pi\alpha)\cos(\tfrac{\alpha\pi}{2}),\\
A_6 &= -\Big(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\Big)^2+(2\mu+\gamma+\eta)^2+2\mu(\mu+\gamma+\eta)\cos(\pi\alpha),\\
A_7 &= 2\Big(-\mu(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I})^2+\mu(\mu+\gamma+\eta)(2\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}),\\
A_8 &= \mu^2(\mu+\gamma+\eta)^2-\mu^2\Big(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\Big)^2.
\end{aligned}$$

We have

$$\begin{aligned}
A_5 &= 2(2\mu+\gamma+\eta)\sin(\pi\alpha)\sin(\tfrac{\alpha\pi}{2})+2(2\mu+\gamma+\eta)\cos(\pi\alpha)\cos(\tfrac{\alpha\pi}{2}),\\
&= 2(2\mu+\gamma+\eta)\cos(\tfrac{\alpha\pi}{2}),
\end{aligned}$$

which implies that $A_5>0$. Further, we have for A_6

$$\begin{aligned}
A_6 &= -\Big(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\Big)^2+(2\mu+\gamma+\eta)^2+2\mu(\mu+\gamma+\eta)\cos(\pi\alpha),\\
&= (\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2)+\mu^2+2\mu(\mu+\gamma+\eta)+2\mu(\mu+\gamma+\eta)\cos(\pi\alpha),\\
&\geq (\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2)+\mu^2+2\mu(\mu+\gamma+\eta)-2\mu(\mu+\gamma+\eta),\\
&\geq (\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2)+\mu^2.
\end{aligned}$$

Hence, if $\mathcal{R}_0 < 1$ we get $A_6 > 0$. For A_7

$$\begin{aligned} A_7 &= 2\mu\Big(-(e^{-\mu\tau}\tfrac{\partial f(S^0,0)}{\partial I})^2 + (\mu+\gamma+\eta)(2\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}), \\ &= 2\mu\Big(-(e^{-\mu\tau}\tfrac{\partial f(S^0,0)}{\partial I})^2 + (\gamma+\eta+\mu)^2 + \mu(\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}), \\ &= 2\mu\Big((\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2) + \mu(\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}). \end{aligned}$$

If $\mathcal{R}_0 < 1$, consequently $A_7 > 0$. Finally

$$\begin{aligned} A_8 &= \mu^2(\mu+\gamma+\eta)^2 - \mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^0,0)}{\partial I}\Big)^2, \\ &= \mu^2(\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2). \end{aligned}$$

If $\mathcal{R}_0 < 1$, then $A_8 > 0$. Since $\omega > 0$, we conclude that equation (2.12) cannot have a positive real root, and hence equation (2.10) has no purely imaginary roots. On the other hand, the characteristic equation of the linearised system (2.8) at E^0 is given by

$$\begin{aligned} \begin{vmatrix} -\mu-\lambda & -\frac{\partial f}{\partial I}(S^0,0) \\ 0 & -(\gamma+\eta+\mu)+e^{-\mu\tau}\frac{\partial f}{\partial I}(S^0,0)-\lambda \end{vmatrix} &= (-\mu-\lambda)\Big(e^{-\mu\tau}\tfrac{\partial f}{\partial I}(S^0,0)-(\gamma+\mu+\eta)-\lambda\Big), \\ &= (\lambda+\mu)\,(\lambda-(\mathcal{R}_0-1))\,(\gamma+\mu+\eta), \\ &= 0, \end{aligned}$$

which have two negative real roots $\lambda_1 = -\mu < 0$ and $\lambda_2 = \mathcal{R}_0 - 1 < 0$ if < 1. The condition $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$ is then satisfied, by Lemma1.2.8 /(1), the free steady state is locally asymptotically stable.

2.3.2 Local stability of endemic steady state

Next, we shall show that the endemic steady state is locally stable EE^*.

Theorem 2.3.2 *Suppose that $\tau = 0$. If $\mathcal{R}_0 > 1$, thus the endemic steady sate EE^* is locally asymptotically stable if*

$$\frac{\partial f}{\partial I}(S^*, I^*) < (\mu+\gamma+\eta). \tag{2.13}$$

Proof 2.3.3 *Put $\tau = 0$. The characteristic equation of system (2.8) at EE^* adopts the form*

$$\lambda^2 + b\lambda + c = 0, \tag{2.14}$$

where

$$\begin{aligned} b &= \left[\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu + \gamma + \eta - \frac{\partial f(S^*,I^*)}{\partial I}\right], \\ c &= \mu\left[(\mu+\gamma+\eta) - \frac{\partial f(S^*,I^*)}{\partial I}\right] + (\mu+\gamma+\eta)\frac{\partial f(S^*,I^*)}{\partial S}. \end{aligned}$$

it is evident that from hypothesis (H_3) *that* $b > 0, c > 0$ *and the Routh-Hurwitz criterion imply that every roots* λ *of (2.14) have negative real parts, which means that* $|\arg(\lambda)| > \frac{\pi}{2} > \alpha\frac{\pi}{2}$. *By Lemma 1.2.8 /(2) the endemic steady state* EE^* *is locally asymptotically stable.*

Theorem 2.3.3 *Suppose that* $\tau > 0$. *If the basic reproduction number, then the endemic steady state* EE^* *is locally asymptotically stable if condition (2.13) holds.*

Proof 2.3.4 *To prove this theorem, we use similar arguments as in Theorem 2.3.1. from (2.9) the characteristic equation at* E^* *is as following*

$$\begin{aligned} & s^{2\alpha} + \left[\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu - e^{-(\mu+s)\tau}\frac{\partial f(S^*,I^*)}{\partial I} + \gamma + \eta\right]s^{\alpha} + \\ & (\mu+\gamma+\eta)\left(\mu + \frac{\partial f(S^*,I^*)}{\partial S}\right) - \mu e^{-(\mu+s)\tau}\frac{\partial f(S^*,I^*)}{\partial I} = 0. \end{aligned} \tag{2.15}$$

Assume that the equation (2.15) has a purely imaginary root $s = we^{i\frac{\pi}{2}}$, $w > 0$. *Substituting s in equation (2.15) and separating real and imaginary parts, we find*

$$\begin{cases} B_1\cos(w\tau) + B_2\sin(w\tau) &= B_3, \\ B_2\cos(w\tau) - B_1\sin(w\tau) &= B_4, \end{cases} \tag{2.16}$$

here

$$\begin{aligned} B_1 &= \mu e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I} + e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}w^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right), \\ B_2 &= w^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}, \\ B_3 &= \omega^{2\alpha}\cos(\alpha\pi) + \omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\left(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu + \gamma + \eta\right) \\ & \quad + \left(\mu + \frac{\partial f(S^*,I^*)}{\partial S}\right)(\mu+\gamma+\eta), \\ B_4 &= \omega^{2\alpha}\sin(\alpha\pi) + \omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\left(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu + \gamma + \eta\right). \end{aligned}$$

Adding the squares of both equations (2.16) gives

$$w^{4\alpha} + B_5 w^{3\alpha} + B_6 w^{2\alpha} + B_7 w^{\alpha} + B_8 = 0, \tag{2.17}$$

where

$$\begin{aligned}
B_5 &= 2\cos(\alpha\pi)\cos(\tfrac{\alpha\pi}{2})\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big)\\
&\quad +2\sin(\alpha\pi)\sin(\tfrac{\alpha\pi}{2})\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big),\\
B_6 &= \Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big)^2-\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&\quad +2\Big(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)\cos(\alpha\pi)(\mu+\gamma+\eta),\\
B_7 &= 2\Big(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)\cos(\tfrac{\alpha\pi}{2})\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big)(\mu+\gamma+\eta)\\
&\quad -2\mu\cos(\tfrac{\alpha\pi}{2})\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2,\\
B_8 &= \Big(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)^2(\mu+\gamma+\eta)^2-\mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2.
\end{aligned}$$

It is evident that if all the coefficients B_i $(i=5,\ldots,8)$ are positive. Thus, the equation (2.17) cannot have a positive root. Since

$$\begin{aligned}
B_5 &= 2\cos(\alpha\pi)\cos(\tfrac{\alpha\pi}{2})\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big)\\
&\quad +2\sin(\alpha\pi)\sin(\tfrac{\alpha\pi}{2})\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big),\\
&= 2\cos(\tfrac{\alpha\pi}{2})\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big),
\end{aligned}$$

thus, $B_5>0$. And we have

$$\begin{aligned}
B_6 &= \Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big)^2-\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&\quad +2(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})\cos(\alpha\pi)(\mu+\gamma+\eta),\\
&> \Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta\Big)^2-\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&\quad -2(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})(\mu+\gamma+\eta),\\
&= \Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+\mu\Big)^2+\Big(\mu+\gamma+\eta\Big)^2-\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2,\\
&= \Big(\mu+\gamma+\eta-e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)\Big(\mu+\gamma+\eta+e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)\\
&\quad +\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+\mu\Big)^2,
\end{aligned}$$

using the condition (2.13), we get $B_6 > 0$. Further,

$$\begin{aligned} B_7 &= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})(\tfrac{\partial f(S^*,I^*)}{\partial S}+2\mu+\gamma+\eta)(\mu+\gamma+\eta) \\ &\quad -\mu\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2\Big], \\ &= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})^2(\mu+\gamma+\eta) \\ &\quad +(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})(\mu+\gamma+\eta)^2-\mu\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2\Big], \\ &= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})^2(\mu+\gamma+\eta)+\tfrac{f(S^*,I^*)}{\partial S}(\mu+\gamma+\eta)^2 \\ &\quad +\mu(\mu+\gamma+\eta)^2-\mu\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2\Big], \\ &= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})^2(\mu+\gamma+\eta)+\tfrac{f(S^*,I^*)}{\partial S}(\mu+\gamma+\eta)^2 \\ &\quad +\mu\Big((\mu+\gamma+\eta-e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I})(\mu+\gamma+\eta+e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I})\Big)\Big], \end{aligned}$$

and

$$\begin{aligned} B_8 &= (\mu+\tfrac{\partial f(S^*,I^*)}{\partial S})^2\Big(\mu+\gamma+\eta\Big)^2-\mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2, \\ &= \Big((\tfrac{\partial f(S^*,I^*)}{\partial S})^2+2\mu\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)(\mu+\gamma+\eta)^2 \\ &\quad +\mu^2(\mu+\gamma+\eta)^2-\mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2, \\ &= \Big((\tfrac{\partial f(S^*,I^*)}{\partial S})^2+2\mu\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)(\mu+\gamma+\eta)^2 \\ &\quad +\mu^2\Big(\mu+\gamma+\eta-e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)\Big(\mu+\gamma+\eta+e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big). \end{aligned}$$

Condition (2.13) yields to $B_7 > 0$ and $B_8 > 0$. Then, we arrive at the conclude that equation (2.17) cannot have positive roots. The characteristic equation of system (2.4) at EE^* takes the form

$$\begin{aligned} P(\lambda) &= \begin{vmatrix} -\Big(\lambda+\tfrac{\partial f(S^*,I^*)}{\partial S}+\mu\Big) & -\tfrac{\partial f(S^*,I^*)}{\partial I} \\ e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial S} & -(\gamma+\mu+\eta)+e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}-\lambda \end{vmatrix}, \\ &= \lambda^2+\lambda\Big((\gamma+\eta+\mu)-e^{-\mu\tau}\frac{\partial f}{\partial I}(S^*,I^*)+\frac{\partial f}{\partial S}(S^*,I^*)+\mu\Big) \\ &\quad +\Big(\tfrac{\partial f(S^*,I^*)}{\partial S}+\mu\Big)\Big((\gamma+\eta+\mu)-e^{-\mu\tau}\frac{\partial f}{\partial I}(S^*,I^*)\Big)+\frac{\partial f}{\partial S}(S^*,I^*)\frac{\partial f}{\partial I}(S^*,I^*)e^{-\mu\tau}, \\ &= 0. \end{aligned}$$

According to condition (2.13), all the coefficients of P are positive, and by the Routh-Hurewitz theorem, all the roots have negative real parts. As a result, the condition $|\arg(\lambda)| >$

$\alpha\frac{\pi}{2}$ is satisfied. By Lemma 1.2.8/1, the endemic steady state EE^* is locally asymptotically stable.

2.4 Global stability

In this part, we apply the Lyapunov function approach to analyze the global stability of both free and endemic stable states. Initially, we establish the free steady state's global stability.

2.4.1 Global stability of free steady state

Theorem 2.4.1 *If the basic reproduction number less then one, then the free steady state E^0 is globally asymptotically stable.*

Proof 2.4.1 *As stated below, define the Lyapunov function V*

$$V(t) \quad = \quad S(t) - S^0 - \int_0^t \frac{f(S^0,I(t))}{f(S(\theta),I(t))} d\theta + e^{\mu\tau} I(t) + \int_{t-\tau}^t f(S(\theta),I(\theta)) d\theta,$$

clearly V is non-negative defined function at E^0. Then

$$D^\alpha V(t) \leq \left(1 - \frac{f(S^0,I(t))}{f(S(t),I(t))}\right) D^\alpha S(t) + e^{\mu\tau} D^\alpha I(t) + f(S(t),I(t)) - f(S(t-\tau),I(t-\tau)).$$

Through application of the two system equations (2.4), we find

$$\begin{aligned} D^\alpha V(t) \quad \leq \quad & \left(1 - \frac{f(S^0,I(t))}{f(S(t),I(t))}\right)\Big(\lambda - \mu S(t) - f(S(t),I(t))\Big) + f(S(t-\tau, I(t-\tau)) \\ & - e^{\mu\tau}(\mu + \gamma + \eta) I(t) + f(S(t),I(t)) - f(S(t-\tau),I(t-\tau)), \end{aligned}$$

since $\lambda = \mu S^0$, then

$$D^\alpha V(t) \leq \mu\left(1 - \frac{f(S^0,I(t))}{f(S(t),I(t))}\right)(S^0 - S(t)) + f(S^0,I(t)) - (\mu + \eta + \gamma) e^{\mu\tau} I(t),$$

the condition $\frac{\partial^2 f(S(t),I(t))}{\partial^2 I(t)} < 0$ yields that

$$f(S^0,I(t)) < I(t)\frac{\partial f(S^0,I(t))}{\partial I(t)} < I(t)\frac{\partial f(S^0,0)}{\partial I(t)},$$

and hence

$$\begin{aligned} D^{\alpha}V(t) &\leq \mu\Big(1-\frac{f(S^0,I(t))}{f(S(t),I(t))}\Big)(S^0-S(t))+I(t)\frac{\partial f(S^0,0)}{\partial I(t)}-(\mu+\eta+\gamma)e^{\mu\tau}I(t), \\ &\leq \mu\Big(1-\frac{f(S^0,I(t))}{f(S(t),I(t))}\Big)(S^0-S(t))+(\mu+\eta+\gamma)e^{\mu\tau}(\mathcal{R}_0-1)I(t). \end{aligned}$$

In the first variable, we observe that f is an increasing function.

$$\frac{f(S^0,I(t))}{f(S(t),I(t))}\geq 1,\ \forall\ S^0\geq S(t),$$
$$\frac{f(S^0,I(t))}{f(S(t),I(t))}\leq 1,\ \forall\ S^0\leq S(t),$$

so,

$$\Big(1-\frac{f(S^0,I(t))}{f(S(t),I(t))})(S^0-S(t)\Big)\leq 0.$$

If $\mathcal{R}_0<1$, thus

$$D^{\alpha}V(t)\leq -W_3,$$

here

$$W_3=\mu(1-\frac{f(S^0,I(t))}{f(S(t),I(t))})(S(t)-S^0)+(\mu+\eta+\gamma)e^{\mu\tau}(1-\mathcal{R}_0)I(t)\geq 0.$$

According to Lemma 1.2.4, and since the free steady state E^0 is locally asymptotically stable, then by the LaSalle invariance principle [24], E^0 is globally asymptotically stable.

2.4.2 Global stability of endemic steady state

The global stability of the endemic stable state is examined in the following theorem.

Theorem 2.4.2 *Assume that $\mathcal{R}_0>1$ and the condition (2.13) holds. Then, the endemic steady state is globally asymptotically stable if the following inequality is true*

$$\Big(\frac{f(S^*,I^*)}{f(S,I^*)}-\frac{f(S^*,I)}{f(S,I)}\Big)\Big(1-\frac{f(S,I)}{f(S^*,I^*)}\Big)\leq 0,\quad \forall S,I>0.$$

Proof 2.4.2 *Let $H(x)=x-1-\ln x$. Note that $H:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ has a strict global minimum at $x=1$.*

Let Lyapunov function L be as follows

$$\begin{aligned} L(t) &= S(t) - S^* - \int_{S^*}^{S(t)} \frac{f(S^*,I(t))}{f(\theta,I(t))} d\theta + e^{\mu\tau}\Big(I(t) - I^* - \int_{I^*}^{I(t)} \frac{f(S(t),I^*)}{f(S(t),\theta)} d\theta\Big) \\ &+ \iota \int_0^{\tau} \Big(\frac{f(S(t-\theta),I(t-\theta))}{f(S^*,I^*)} - 1 - \ln \frac{f(S(t-\theta),I(t-\theta))}{f(S^*,I^*)}\Big) d\theta, \end{aligned}$$

where

$$\iota = f(S^*, I^*).$$

The fractional derivative of L verify

$$\begin{aligned} D^{\alpha} L(t) &\leq \Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big) D^{\alpha} S(t) + e^{\mu\tau}\Big(1 - \frac{f(S(t),I^*)}{f(S(t),I(t))}\Big) D^{\alpha} I(t) \\ &+ \iota\Big[- \frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)} + \frac{f(S(t),I(t))}{f(S^*,I^*)} + \ln\Big(\frac{f(S(t-\tau),I(t-\tau)}{f(S(t),I(t))}\Big)\Big], \end{aligned} \tag{2.18}$$

by utilizing the system's initial equation (2.4), we get

$$\Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big) D^{\alpha} S(t) = \Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)(\lambda - \mu S(t) - f(S(t), I(t))).$$

According to the system's initial equation (2.5),

$$\lambda = \mu S^* + f(S^*, I^*),$$

that is,

$$\Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big) D^{\alpha} S(t) = \Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(\mu(S^* - S(t)) + f(S^*, I^*) - f(S(t), I(t))\Big),$$

thus,

$$\begin{aligned} \Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big) D^{\alpha} S(t) &= \mu S^*\Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)(1 - \frac{S(t)}{S^*}) \\ &+ f(S^*, I^*)\Big(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1 - \frac{f(S(t),I(t))}{f(S^*,I^*)}\Big). \end{aligned} \tag{2.19}$$

Since

$$e^{\mu t}\Big(1 - \frac{f(S(t),I^*)}{f(S(t),I(t))}\Big) D^{\alpha} I(t) = \Big(1 - \frac{f(S(t),I^*)}{f(S(t),I(t))})\Big(f(S(t-\tau), I(t-\tau)) - e^{\mu\tau}(\mu + \gamma + \eta)I(t)\Big),$$

using the second equation of system (2.4), we find

$$\frac{f(S^*,I^*)}{I^*} = (\mu+\eta+\gamma)e^{\mu\tau},$$

which leads to,

$$e^{\mu t}\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)D^\alpha I(t) \quad = \quad f(S^*,I^*)\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\frac{I(t)}{I^*}\Big),$$

hence,

$$\begin{aligned} e^{\mu t}\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)D^\alpha I(t) \quad &= \quad f(S^*,I^*)\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{I(t)}{I^*}\Big) \\ &+f(S^*,I^*)\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\frac{f(S(t),I(t))}{f(S(t),I^*)}\Big). \end{aligned} \tag{2.20}$$

Substituting the equations (2.19) and (2.20) into equation (2.18), we get

$$\begin{aligned} D^\alpha L(t) \quad \leq \quad & \mu S^*\Big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)(1-\frac{S(t)}{S^*}) \\ &+f(S^*,I^*)\Big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\Big) \\ &+f(S^*,I^*)\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{I(t)}{I^*}\Big) \\ &+f(S^*,I^*)\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\frac{f(S(t),I(t))}{f(S(t),I^*)}\Big) \\ &+f(S^*,I^*)\Big[-\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}+\frac{f(S(t),I(t))}{f(S^*,I^*)}+\ln\frac{f(S(t-\tau),I(t-\tau))}{f(S(t),I(t))}\Big]. \end{aligned} \tag{2.21}$$

Put

$$\begin{aligned} A \quad = \quad & \Big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\Big) \\ &+\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\frac{f(S(t),I(t))}{f(S(t),I^*)}\Big) \\ &+\Big[-\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}+\frac{f(S(t),I(t))}{f(S^*,I^*)}+\ln\frac{f(S(t-\tau),I(t-\tau))}{f(S(t),I(t))}\Big], \end{aligned}$$

then (2.21) becomes

$$\begin{aligned} D^\alpha L(t) \quad \leq \quad & \mu S^*\Big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)(1-\frac{S(t)}{S^*}) \\ &+f(S^*,I^*)\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{I(t)}{I^*}\Big)+Af(S^*,I^*), \end{aligned} \tag{2.22}$$

and after some calculations

$$\begin{aligned}
A &= 2-\frac{f(S^*,I(t))}{f(S(t),I(t))}+\frac{f(S^*,I(t))}{f(S^*,I^*)}-\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\\
&\quad+\ln(\frac{f(S(t-\tau),I(t-\tau)}{f(S(t),I(t))}),\\
&= 2-\frac{f(S^*,I(t))}{f(S(t),I(t))}+\frac{f(S^*,I(t))}{f(S^*,I^*)}-\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\\
&\quad+\ln(\frac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)})+\ln(\frac{f(S^*,I^*)}{f(S(t),I^*)})+\frac{f(S^*,I^*)}{f(S(t),I^*)}-\frac{f(S^*,I^*)}{f(S(t),I^*)},\\
&= -\frac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}+1+\ln(\frac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)})\\
&\quad-\frac{f(S^*,I^*)}{f(S(t),I^*)}+1+\ln(\frac{f(S^*,I^*)}{f(S(t),I^*)})-\frac{f(S^*,I(t))}{f(S(t),I(t))}+\frac{f(S^*,I(t))}{f(S^*,I^*)}\\
&\quad-\frac{f(S(t),I(t))}{f(S(t),I^*)}+\frac{f(S^*,I^*)}{f(S(t),I^*)},\\
&= -H\Big(\frac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)-H\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}\Big)-\frac{f(S^*,I(t))}{f(S(t),I(t))}+\frac{f(S^*,I(t))}{f(S^*,I^*)}\\
&\quad-\frac{f(S(t),I(t))}{f(S(t),I^*)}+\frac{f(S^*,I^*)}{f(S(t),I^*)},
\end{aligned}$$

but

$$\begin{aligned}
&-\frac{f(S^*,I(t))}{f(S(t),I(t))}+\frac{f(S^*,I(t))}{f(S^*,I^*)}-\frac{f(S(t),I(t))}{f(S(t),I^*)}+\frac{f(S^*,I^*)}{f(S(t),I^*)}\\
&=\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\Big),
\end{aligned}$$

thus

$$\begin{aligned}
A &= -H\Big(\frac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)-H\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}\Big)\\
&\quad+\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\Big),
\end{aligned} \tag{2.23}$$

changing (2.23) into (2.22), we get

$$\begin{aligned}
D^\alpha L(t) &\le \mu S^*\Big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)(1-\frac{S(t)}{S^*})\\
&\quad+f(S^*,I^*)\Big\{\Big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{I(t)}{I^*}\Big)\\
&\quad-H\Big(\frac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)-H\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}\Big)\\
&\quad+\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\Big)\Big\}.
\end{aligned} \tag{2.24}$$

But f is a monotonically increasing function with respect to S, then

$$\begin{aligned}
&\frac{f(S^*,I(t))}{f(S(t),I(t))}\ge 1,\ \forall\ S^*\ge S(t),\\
&\frac{f(S^*,I(t))}{f(S(t),I(t))}\le 1,\ \forall\ S^*\le S(t),
\end{aligned}$$

then

$$\Big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\frac{S(t)}{S^*}\Big)\le 0,$$

using the hypothesis (H_4), we find

$$\left(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\right)\left(\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{I(t)}{I^*}\right)\leq 0.$$

If

$$\left(\frac{f(S^*,I^*)}{f(S(t),I^*)}-\frac{f(S^*,I(t))}{f(S(t),I(t))}\right)\left(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\right)\leq 0,$$

thus

$$D^\alpha L(t)\leq -W_3,$$

here

$$\begin{aligned} W_3 \;=\;& \mu S^*\Big(\frac{f(S^*,I(t))}{f(S(t),I(t))}-1\Big)(1-\frac{S(t)}{S^*}) \\ &+f(S^*,I^*)\Bigg\{\Big(\frac{f(S(t),I^*)}{f(S(t),I(t))}-1\Big)\Big(\frac{f(S(t),I(t))}{f(S(t),I^*)}-\frac{I(t)}{I^*}\Big) \\ &+H\Big(\frac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)+H\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}\Big) \\ &+\Big(\frac{f(S^*,I^*)}{f(S(t),I^*)}-\frac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(\frac{f(S(t),I(t))}{f(S^*,I^*)}-1\Big)\Bigg\}\geq 0. \end{aligned}$$

By the LaSalle invariance principle and Lemma 1.2.4, we conclude that the endemic steady state is globally asymptotically stable.

Remark 2.4.1 *The condition*

$$\left(\frac{f(S^*,I^*)}{f(S,I^*)}-\frac{f(S^*,I)}{f(S,I)}\right)\left(1-\frac{f(S,I)}{f(S^*,I^*)}\right)\leq 0,$$

is met by the most often utilized incidence functions, including $f(S,I)=\beta S^n I^m$ and $f(S,I)=\frac{\beta S^n I^m}{1+aS}$ with $a>0$ and $n,m\geq 0$. More generally, the condition is also satisfied by the incidence function of the type $f(S,I)=\frac{\beta SI}{\psi(I)}$, where ψ is a concave function.

2.5 Numerical simulations

In this section, we describe the numerical scheme for solving a fractional-order delayed equation. A modified Adams-Bashforth-Moulton method is proposed in [3] to include both fractional-order and delay. Take a look at the following equation.

$$\begin{cases} D^{\alpha} y(t) = f\left(t, y(t), y(t-\tau)\right), & t \in [0, T] \\ y(t) = g(t), & t \in [-\tau, 0], \end{cases} \tag{2.25}$$

where $0 < \alpha \leq 1$. Let the uniform grid $\{t_n = nh : n = -k, -k+1, \cdots, -1, 0, 1, \cdots, N\}$ where k and N are integers such that $h = \frac{T}{N} = \frac{\tau}{k}$. We let

$$y_h(t_j) = g(t_j), \quad j = -k, -k+1, \cdots, 0$$

with

$$y_h(t_j - \tau) = y_h(jh - kh) = y_h(t_{j-k}), \quad j = 0, 1, \cdots, N$$

where $y_h(t_n)$ is an approximation of $y(t_n)$. The scheme takes the form

$$\begin{aligned} y_h(t_{n+1}) &= g(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left(f\left(t_{n+1}, y_h^p(t_{n+1}), y_h(t_{n+1-k})\right)\right) \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, y_h(t_j), y_h(t_{j-k})\right), \end{aligned} \tag{2.26}$$

here

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \leq j \leq n \\ 1 & j = n+1 \end{cases}$$

The predictor term $y_h^p(t_{n+1})$ which appears on the right hand side of (2.26) is evaluated by the expression

$$y_h^p(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, y_h(t_j, y_h(t_{j-k}))\right), \tag{2.27}$$

where

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha} \left((n+1-j)^{\alpha} - (n-j)^{\alpha} \right).$$

The scheme (2.26)-(2.27) is convergent of order $p = \min(2, 1+\alpha)$ (see [3]).
As an example, we take the incidence function $f(S, I) = \beta S I$ which leads to the following

model

$$\begin{cases} D^\alpha S(t) &= \lambda - \beta S(t)I(t) - \mu S(t), \\ D^\alpha E(t) &= \beta S(t)I(t) - \beta S(t-\tau)I(t-\tau)e^{-\mu\tau} - \mu E(t), \\ D^\alpha I(t) &= \beta S(t-\tau)I(t-\tau)e^{-\mu\tau} - (\mu+\eta+\gamma)I(t), \\ D^\alpha R(t) &= \gamma I(t) - \mu R(t). \end{cases} \tag{2.28}$$

Utilizing the SEIR model (2.28), the illness that COVID-19 created in Algeria during the beginning of the infection is explained. We then start our simulations from February 25, 2020, according to sources indicating that the first instance of the COVID-19 epidemic in Algeria happened on that day [38]. In this instance, the basic reproduction number is $\mathcal{R}_0 = \beta \frac{e^{-\mu\tau}}{\mu+\eta+\gamma}\frac{\lambda}{\mu}$. We may presume that the death rate mortality of infected people is $\eta = 0$ since our simulations are evaluate over a brief time period (a few months). Table 1 provides the system's (2.28) parameters, which are derived from [32].

Table 2.1: Parameters and values of model (2.28).

Parameters	**meaning**	**values**
λ	recruitment rate of susceptible individuals	10^{-5}
β	transmission rate per infectious individuals	2.1
μ	Death rate of all individuals	10^{-5}
γ	transfer rate of infected individuals to recovery compartment	0.2
η	Death rate of infected individuals caused by the disease	0
τ	Incubation period	$2-14$ (days)

Source: [32]

We first consider the case with classic derivative $\alpha = 1$ and without delay $\tau = 0$. Numerical simulations give the graphics in figure 2.5. As described in [32], we observe a peak of infected individuals after 90 days from the beginning of the infection and 80% of the population will recover. In figure 2.5, we have plotted the solutions of system (2.28) in the case $\alpha = 0.9$ and $\tau = 5$ days. We can observe a peak of infected individuals of around 140 days after the beginning of the infection which corresponds to late July 2020. In figure 2.4, we have plotted the solutions in the case $\tau = 8$ days and $\alpha = 0.9$ and in figure 2.5, we have taken $\tau = 10$ days. We first observe that the recovered populations grows quickly whereas the infected and exposed populations

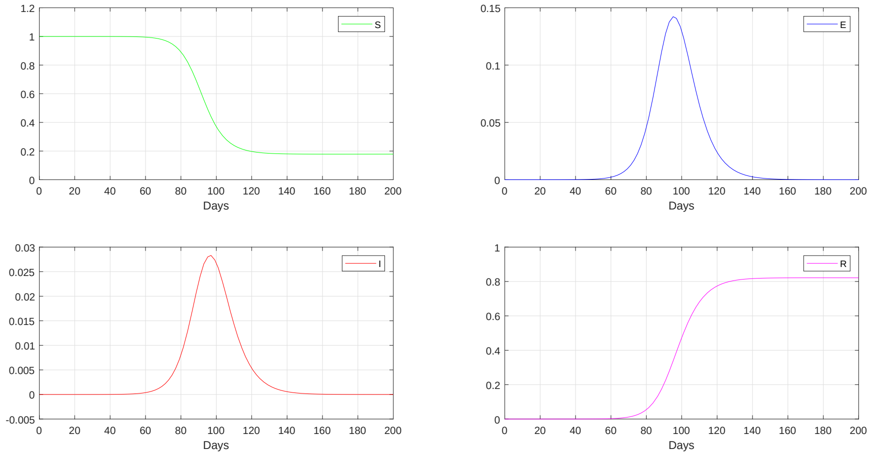


Figure 2.2: Model (2.28) in the case $\alpha = 1$ and without delay $\tau = 0$.

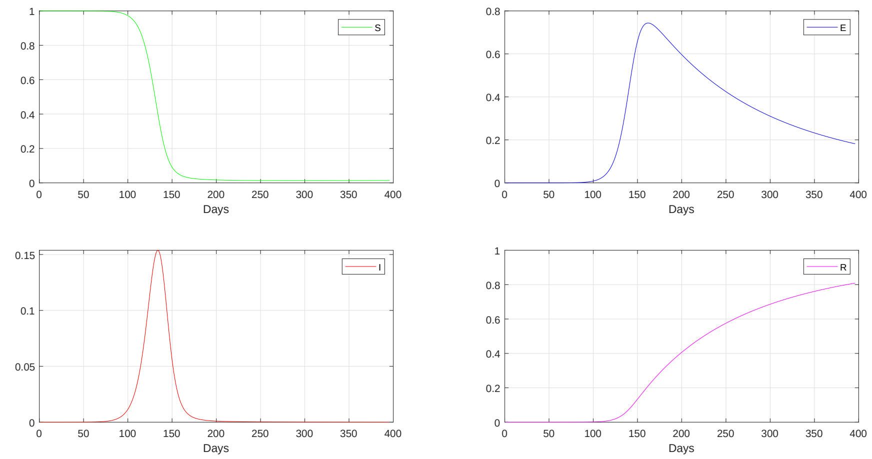


Figure 2.3: Solutions of model (2.28) in the case $\tau = 5$ and $\alpha = 0.9$.

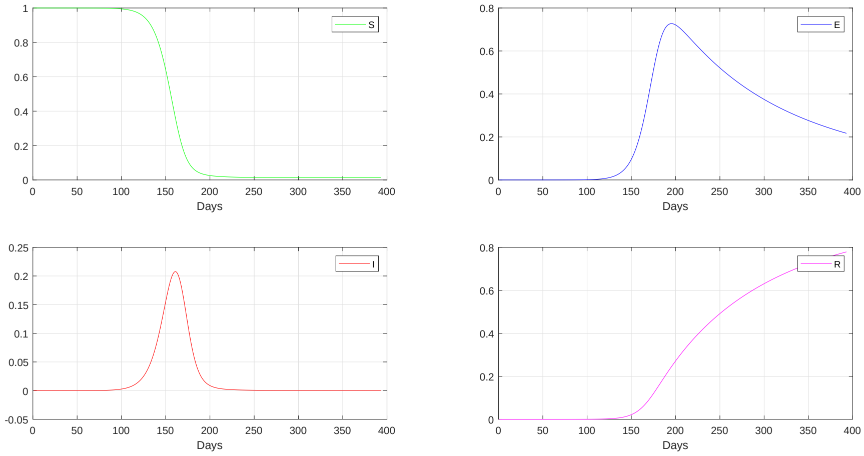


Figure 2.4: Solutions of model (2.28) in the case $\tau = 8$ days.

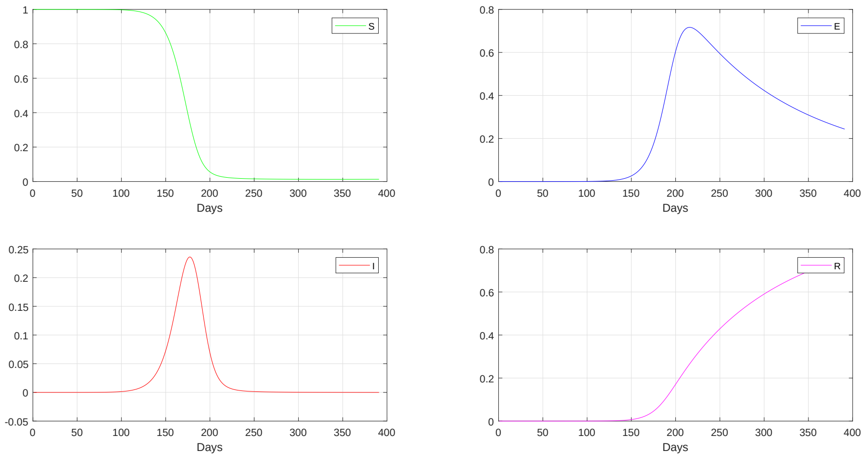


Figure 2.5: Solutions of model (2.28) in the case $\tau = 10$ days.

decrease significantly which means that the majority of the populations will recover. Further, we observe a peak of infected individuals which grows with the delay τ of the incubation period. For a value of τ around $8 \sim 10$ days this peak appears after a period of 160 days from the beginning of the infection which corresponds to late July. In figure 8, we give a graphic published by WHO[49] which shows a peak of new infected individuals of COVID-19 in Algeria at 24 July 2020. We can conclude that the model (2.28) with fractional derivative and time delay describes the outbreak of COVID-19 more precisely than the one with classic derivative and without delay.

In figure 2.5, we have plotted the behaviour of compartments S, E, I, R for different

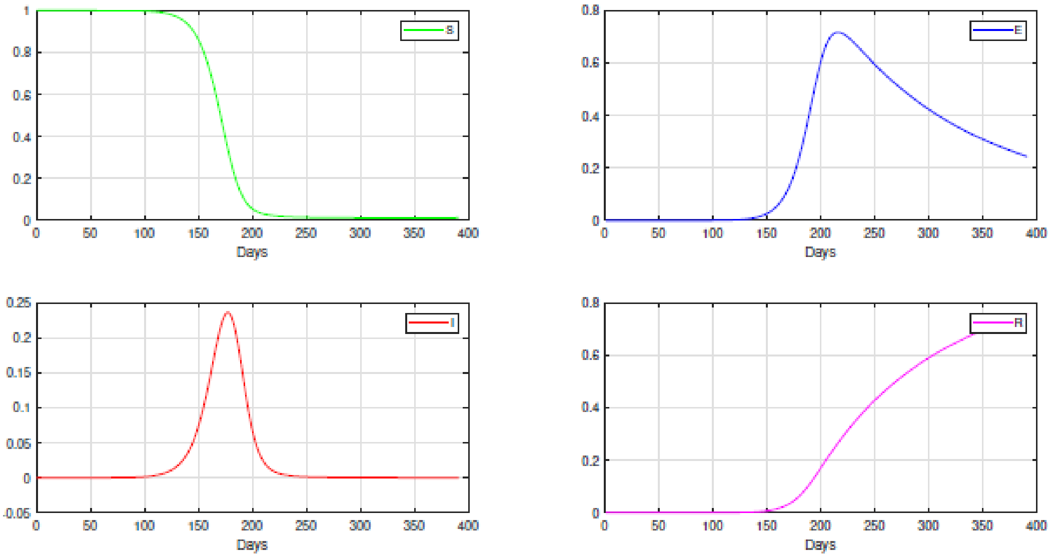


Figure 2.6: Behaviour of S, E, I, R for different values of γ: we have taken $\tau = 8$ and $\beta = 2.1$

values of γ. Since $\frac{1}{\gamma}$ is the average time in compartment I before isolation, the value $\gamma = 1$ means that almost all infected individuals are asymptomatic and recover after on average one day. The value $\gamma = 0.2$ correspond to an average period of around 5 days in infectious compartment and appears more plausible [32]. The plots in figure 6 show that this case lead to the infection of 90% of population in Algeria. We think that the real value of γ is around 0.2.

Finally, in figures 2.5 we have plotted the solutions of susceptible, exposed, infected and recovered individuals for the values of $\alpha = 0.6, 0.7, 0.8, 0.9$. We can observe that when α is decreased the amount of susceptible and infected individuals will be reduced. As in [7] we conclude that small memory of the infection effect maximizes the number of COVID-19 healthy individuals.

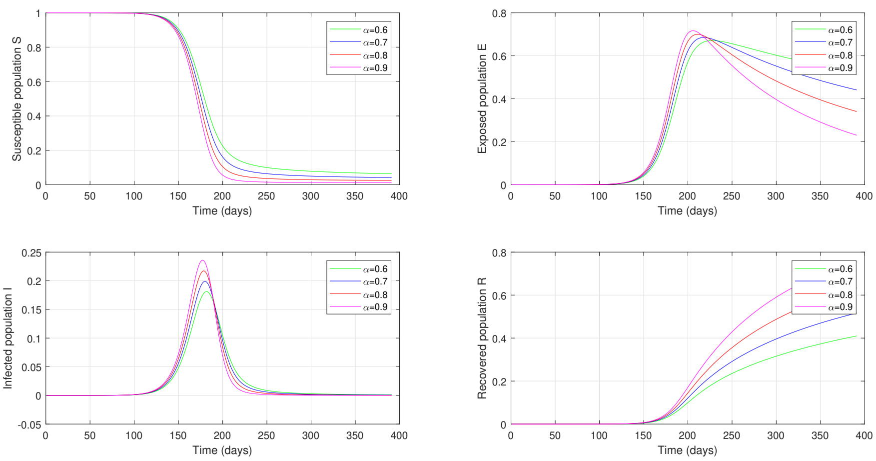
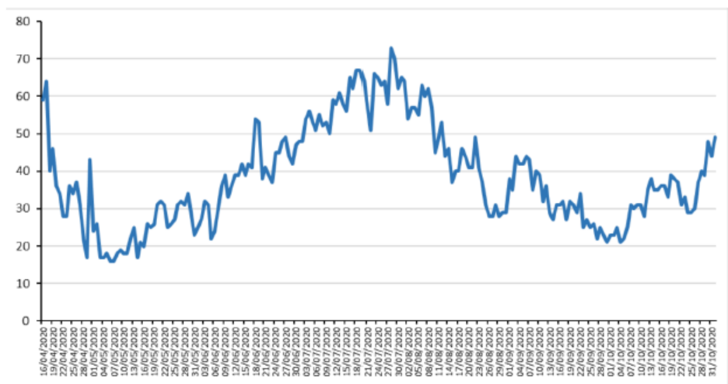


Figure 2.7: Behaviour of the compartments S, E, I, R of model (2.28) with different values of α



Number of new infected individuals in Algeria from April to October 2020 [49, 38]

Chapter 3

Stability Behavior of a $SEIR$ Epidemic Model with More General Incidence Rate

3.1 Model formulated

In reality, the exposed person can contract the majority of epidemics, including not only the infected class in most of epidemic such as Covid-19, HIV/AIDS, measles..., but the previous study ignore this possibility. In addition, the incidence function gives more information of the disease transmission. In litterature, there are many kind of incidence function for example the bilinear incidence function βSI where $\beta > 0$[46], the saturated incidence [29], the nonmonotone incidence rates ...But this function can not given a detail information of the of infectious pandemic because can change with the surrounding environment.

Motivated of this reasons, we think this factors in our study and formulated fractional non linear system when the disease incidence a more general nonlinear incidence form, can be calculated as $\beta_E(E)g(E)h(S)$, $\beta_I(I)f(I)h(S)$, when the population is divided in four compartments, namely, the susceptible S, exposed E, infected I and recovered R, $\beta_E(E)$ represent the transmission coefficient between compartments S and E, $\beta_I(I)$ represent the transmission coefficient between compartments S and I when $\beta_E \geq 0, \beta_I \geq$

0. The model takes the form

$$
\begin{cases}
D^{\alpha}S(t) &=& \lambda - \Big[\beta_E(E(t))g(E(t)) + \beta_I(I(t))f(I(t))\Big]h(S(t)) - \mu S(t), \\
D^{\alpha}E(t) &=& \Big[\beta_E(E(t))g(E(t)) + \beta_I(I(t))f(I(t))\Big]h(S(t)) - (\delta+\mu)E(t), \\
D^{\alpha}I(t) &=& \delta E(t) - (\gamma+\mu)I(t), \\
D^{\alpha}R(t) &=& \gamma I(t) - \mu R(t),
\end{cases}
\tag{3.1}
$$

with the following initial conditions

$$
\begin{aligned}
&S(0) = S_0 \geq 0,\ E(0) = E_0 \geq 0, \\
&I(0) = I_0 \geq 0,\ R(0) = R_0 \geq 0,
\end{aligned}
\tag{3.2}
$$

where D^{α} denote the fractional order derivative with respect to time t. We assume that $g, f, h, \beta_E, \beta_I : \mathbb{R}^{+} \to \mathbb{R}^{+}$ are a twice differential nonlinear function satisfying the following hypotheses :

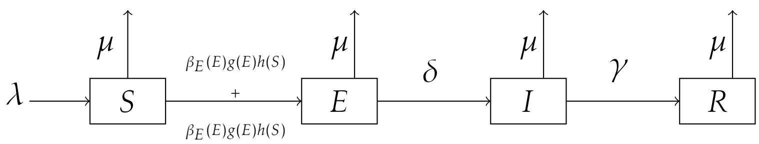
(A_1) g, f, and h are all positive and only vanish at 0.

(A_2) g, f and h are monotonically increasing.

(A_3) β_E and β_I are monotonically decreasing with $\beta_E(E) < \beta_E(0)$, $\beta_I(I) < \beta_I(0)$, and $\beta_E(0) = \beta_I(0) = 0$.

(A_4) f and g are concave functions.

The properties $(A_1), (A_2)$, for the functions f, g and h include common incidence functions (see. [19][29][39]). All parameter values are assumed to be non-negative, λ is the influx of the susceptible individual, μ the death rate, δ the transmission coefficient between E to I, γ the transmission coefficient between I to R. The transfer diagram of system (3.1) is given by



Noting by

Figure 3.1: The diagram transfer of system (3.1).

$$N(t) = S(t) + E(t) + I(t) + R(t),$$

the total number the population at time t, we can see from the previous system that the last equation is independent from the three first ones. Then, system (3.1) can be reduced to the following one

$$\begin{cases} D^\alpha S(t) = \lambda - \Big[\beta_E(E)g(E) + \beta_I(I)f(I)\Big]h(S) - \mu S, \\ D^\alpha E(t) = \Big[\beta_E(E)g(E) + \beta_I(I)f(I)\Big]h(S) - (\delta + \mu)E, \\ D^\alpha I(t) = \delta E - (\gamma + \mu)I. \end{cases} \tag{3.3}$$

Positivity and boundedness of solution

In the bio-mathematics model (population model), the first question posed is the existence of positive and globally solution. For this reasons, since the dynamical behavior of system (3.1) is equivalent to that of system (3.3), we have the next theorem.

Theorem 3.1.1 *For all non-negative initial data, the solution (S, E, I) of the system (3.3) exist, non-negative and uniformly bounded on $[0, +\infty)$. Furthermore, the feasible region of (3.3) can be written as*

$$\Gamma = \Big\{(S, E, I) \in \mathbb{R}^3_+ \ : \ S \leq S^0, \ 0 \leq S + E + I \leq \tfrac{\lambda}{\mu}\Big\}.$$

Proof 3.1.1 *It is easy to prove the existence and uniqueness by using the fundamental theory of fractional equations [27], we only prove the next one. Let $(S,\ E,\ I) \in \mathbb{R}^3_+$, be a solution of system (3.3). If $(0, E, I) \in \mathbb{R}^3_+$, $D^\alpha S = \lambda \geq 0$. If $(S, 0, I) \in \mathbb{R}^3_+$, $D^\alpha E = \beta_I(I)f(I)h(S) \geq 0$. If $(S, E, 0) \in \mathbb{R}^{3n}_+$, $D^\alpha I = \gamma E \geq 0$. Thus, the solution $(S,\ E,\ I) \in \mathbb{R}^3_+$. In order to show that solution of system (3.3) belongs to $\mathbb{R}^{3+}_*$, let us assume that there exists $t_1 = \inf\{t \geq 0\}$ such that one of $S(t_1)$, $E(t_1)$ or $I(t_1)$ is zero, assume that $S(t_1) = 0$, then*

$$D^\alpha S(t_1) = \lambda > 0,$$

witch implies that $D^\alpha S$ is non negative in $[t_1, t_1 + \varsigma[$, according to Corollary 1.2.1 S is

strictly increasing function, this yields that $S(t_1 - \varsigma) < S(t_1) = 0$, with $\varsigma > 0$, then we obtain a contradiction which implies that $S(t) > 0$ for all non negative time.
We can use a similar argumentation for the proof of strict positive of E and I.
Boundedness of solution :
Summing up the three equations in system (3.3) to obtain

$$D^{\alpha} N(t) = \lambda - \mu N(t),$$

using the lemma 1.2.2, we obtain

$$N(t) \leq (N_0 + \lambda\mu^{-1}) E_{\alpha}(-\mu t^{\alpha}) + \lambda\mu^{-1},$$

when $N_0 = N(0) = S_0 + E_0 + I_0$, thus, $\limsup_{t \longrightarrow \infty} N(t) \leq \frac{\lambda}{\mu}$. From the first equation of system (3.3) we can verify that $S(t) \leq S^0$, then

$$\Gamma = \Big\{ (S, E, I) \in \mathbb{R}^3_+ \ : \ S(t) \leq S^0, \ 0 \leq S + E + I \leq \frac{\lambda}{\mu} \Big\}.$$

3.2 The Basic Reproduction Number and Equilibria

3.2.1 The basic reproduction number

Setting the right hand side of the equations of model (3.3) to zero, when absence of epidemic we find that our model admits always the disease-free equilibrium given by $E^0 = (S^0, 0, 0)^T$, where $S^0 = \frac{\lambda}{\mu}$.
Now, we derive the basic reproduction number of system (3.3) by using the method of next-generation matrix (see [42][43]).

Lemma 3.2.1 *The basic reproduction number denoted $\mathcal{R}_0$ of system (3.3) is given by*

$$\mathcal{R}_0 = \frac{\beta_E(0) g^{'}(0) h(S^0)}{(\mu+\delta)} + \frac{\beta_I(0) f^{'}(0) h(S^0)\delta}{(\mu+\gamma)(\mu+\delta)}.$$

Proof 3.2.1 *Let $X = (E, I)^T$, we can see that*

$$D^{\alpha}X = F - V,$$

where

$$F = \begin{pmatrix} \Big[\beta_E(E)g(E) + \beta_I(I)f(I)\Big]h(S) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} (\delta + \mu)E \\ -\delta E + (\gamma + \mu)I \end{pmatrix},$$

the Jacobian matrices of F and V at E^0 denoted $\mathcal{F}$, $\mathcal{V}$ respectively are given by

$$\mathcal{F} = \begin{pmatrix} \beta_E(0)g^{'}(0)h(S^0) & \beta_I(0)f^{'}(0)h(S^0) \\ 0 & 0 \end{pmatrix},$$

and

$$\mathcal{V} = \begin{pmatrix} \delta + \mu & 0 \\ -\delta & \gamma + \mu \end{pmatrix},$$

it is easy to see that $\mathcal{V}$ non-singular matrix. The basic reproduction number is the spectral radius of the matrix $\mathcal{F}\mathcal{V}^{-1}$ [43], after simple calculation we find

$$\mathcal{R}_0 = \frac{\beta_E(0)g^{'}(0)h(S^0)}{(\mu+\delta)} + \frac{\beta_I(0)f^{'}(0)h(S^0)\delta}{(\mu+\gamma)(\mu+\delta)} \ .$$

3.2.2 The equilibrium point

The equilibrium points are the solution of the following system

$$\begin{cases} \lambda - \Big[\beta_E(E^*)g(E^*) + \beta_I(I^*)f(I^*)\Big]h(S^*) - \mu S^* &= 0, \\ \Big[\beta_E(E^*)g(E^*) + \beta_I(I^*)f(I^*)\Big]h(S^*) - (\delta + \mu)E^* &= 0, \\ \delta E^* - (\gamma + \mu)I^* &= 0. \end{cases} \tag{3.4}$$

At absence of endemic i-e $E = I = 0$, it is clear that the system (3.4) has always the free steady states $E^0 = (\frac{\lambda}{\mu}, 0, 0)^T$. In the other hind, when $E \neq 0, I \neq 0$ solving the system we find that

$$\begin{cases} S^* = \frac{\lambda}{\mu} - \frac{(\gamma+\mu)(\delta+\mu)I^*}{\delta\mu}, \\ E^* = \frac{(\gamma+\mu)I^*}{\delta}, \end{cases}$$

where I^* is the roots of the following equation

$$\mathscr{F}(I) = \frac{1}{I}\Big(\beta_E(\frac{(\gamma+\mu)I}{\delta})g(\frac{(\gamma+\mu)I}{\delta}) + \beta_I(I)f(I)\Big)h(\frac{\lambda}{\mu} - \frac{(\gamma+\mu)(\delta+\mu)I}{\delta\mu}) - \frac{(\gamma+\mu)(\delta+\mu)}{\delta} = 0,$$

using the hypothesis (A_3) and (A_4), we obtain the first and second derivative of $\mathscr{F}$ is negative. Since $\lim_{I\longrightarrow 0^+} \mathscr{F}(I) = \frac{(\gamma+\mu)(\delta+\mu)}{\delta}(\mathcal{R}_0 - 1)$, and $\mathscr{F}(\bar{I}) = \frac{\lambda\delta}{(\gamma+\mu)(\delta+\mu)} < 0$. Obviously, if $\mathcal{R}_0 > 1$ then $\mathscr{F}$ has a unique zero solution in $(0, \bar{I})$, for $I > \bar{I}$ we get $S^* < 0$ witch contradiction.

3.3 Local stability of the equilibrium

3.3.1 Local stability of the free-steady state

In this subsection, the local stability of the free-steady state are demonstrated.

Theorem 3.3.1 *If $\mathcal{R}_0 < 1$ the disease free steady state E^0 is locally asymptotically stable.*

Proof 3.3.1 *The general Jacobian matrix $\mathcal{K}$ takes the form*

$$\mathcal{K} = \begin{pmatrix} -\mu - \kappa_1 h'(S) & -\kappa_2 h(S) & -\kappa_3 h(S) \\ \kappa_1 h'(S) & \kappa_2 h(S) - (\delta+\mu) & \kappa_3 h(S) \\ 0 & \delta & -(\gamma+\mu) \end{pmatrix},$$

where

$$\kappa_1 = \Big(\beta_E(E^*)g(E^*) + \beta_I(I^*)f(I^*)\Big),$$
$$\kappa_2 = (\beta_E'(E^*)g(E^*) + \beta_E(E^*)g'(E^*)\Big),$$
$$\kappa_3 = (\beta_I'(I^*)f(I^*) + \beta_I(I^*)f'(I^*))\Big).$$

Using the hypothesis (A_1), (A_2) and (A_3), we find that at the disease-free equilibrium E^0 the *matrix $\mathcal{K}$ is given as follows*

$$A = \begin{pmatrix} -\mu & -\beta_E(0)g^{'}(0)h(S^0) & -\beta_I(0)f^{'}(0)h(S^0) \\ 0 & \beta_E(0)g^{'}(0)h(S^0) - (\delta+\mu) & \beta_I(0)f^{'}(0)h(S^0) \\ 0 & \delta & -(\gamma+\mu) \end{pmatrix}.$$

Denote by $P(\theta)$ the characteristic polynomial of system (3.3) at E^0, thus

$$\begin{aligned} P(\theta) &= \det(\theta I - A), \\ &= -(\theta+\mu)\Big\{\theta^2 + \Big(\gamma+\mu-\beta_E(0)g^{'}(0)h(S^0)+\delta+\mu\Big)\theta \\ &\quad -\beta_E(0)g^{'}(0)h(S^0)(\gamma+\mu) + (\delta+\mu)(\gamma+\mu) - \beta_I(0)f^{'}(0)h(S^0)\delta\Big\}, \\ &= -(\theta+\mu)\Big\{\theta^2 + \Big(\gamma+\mu-\beta_E(0)g^{'}(0)h(S^0)+\delta+\mu\Big)\theta \\ &\quad +(\delta+\mu)(\gamma+\mu)(1-\mathcal{R}_0)\Big\}, \end{aligned}$$

it is clear that P has three roots, $\theta_1 = -\mu$ and θ_2, θ_3 are determined by the following equation

$$\theta^2 + \Big(\gamma+\mu-\beta_E(0)g^{'}(0)h(S^0)+\delta+\mu\Big)\theta + (\delta+\mu)(\gamma+\mu)(1-\mathcal{R}_0) = 0, \tag{3.5}$$

if $\mathcal{R}_0 < 1$, we conclude that $\gamma+\mu-\beta_E(0)g^{'}(0)h(S^0)+\delta+\mu > 0$ and $(\delta+\mu)(\gamma+\mu)(1-\mathcal{R}_0) > 0$, then the equation (3.5) has two negative real roots [42]. Thus $|arg(\theta)| = \pi > \frac{\pi}{2} > \frac{\alpha\pi}{2}$, according to theorem 1.2.3, the free steady states is locally asymptotically stable.

3.3.2 Local stability of the endemic equilibrium

In the next, the local local stability of the endemic equilibrium is discussed.

Theorem 3.3.2 *If $\mathcal{R}_0 > 1$ the endemic steady state EE^* is locally asymptotically stable.*

Proof 3.3.2 *At endemic steady states EE^*, the characteristic equation gives as*

$$\theta^3 + A_2\theta^2 + A_1\theta + A_0 = 0, \tag{3.6}$$

where

$$\begin{aligned}
A_2 &= \mu+\kappa_1 h^{'}(S)+\gamma+\mu-\kappa_2 h(S)+\delta+\mu,\\
A_1 &= (\gamma+\mu)(\mu+\kappa_1 h^{'}(S))-\kappa_2 h(S)(\mu+\kappa_1 h^{'}(S))+(\delta+\mu)(\mu+\kappa_1 h^{'}(S))\\
&\quad -\kappa_2 h(S)(\gamma+\mu)+(\delta+\mu)(\gamma+\mu)-\kappa_3 h(S)\delta,\\
A_0 &= -\kappa_2 h(S)(\gamma+\mu)(\mu+\kappa_1 h^{'}(S))+(\delta+\mu)(\gamma+\mu)(\mu+\kappa_1 h^{'}(S))\\
&\quad -\kappa_3 h(S)(\mu+\kappa_1 h^{'}(S))\delta-\kappa_2 h(S)\kappa_1 h^{'}(S)(\gamma+\mu)+\kappa_3 h(S)\kappa_1 h^{'}(S)\delta,
\end{aligned}$$

the discriminant of $P(\lambda)$ given in the next form

$$D(P)=18A_0A_1A_2+(A_0A_1)^2-4A_2A_0^3-4A_1^3-27A_0^2.$$

According to [2], we have the following lemma

Lemma 3.3.1 *If $\mathcal{R}_0>1$, the endemic equilibrium is locally asymptotically stable if one the following conditions holds*

i If $D(P)>0$, then the necessary and sufficient condition for the equilibrium steady state to be locally asymptotically stable is $A_2>0$, $A_0>0$, $A_1A_2>A_0$.

ii If $D(P)<0$, $A_2>0$, $A_1>0$, $A_0>0$. Then the endemic equilibrium is locally asymptotically stable if $\alpha<\frac{2}{3}$.

iii If $D(P)<0$, $A_2<0$, $A_1<0$ and $\alpha>\frac{2}{3}$. Then all roots of $D(P)$ satisfy the condition $|arg(\theta_i)|<\frac{\alpha\pi}{2}$ $i=1,2,3$.

Now, we turn to given a sufficient conditions to obtain the Hopf bifurcation of fractional system (3.3), by considering α as bifurcation parameter. Using the result show in [26], the endemic equilibrium is locally asymptotically stable if the eigenvalues λ_i of equation (3.6) are less then $\frac{\alpha\pi}{2}$ and unstable if the eigenvalues are more then $\frac{\alpha\pi}{2}$. Thus, we define the following function

$$H(\alpha) \;=\; \frac{\alpha\pi}{2}-\min_{1\le i\le 3}|arg(\lambda_i)|, i=1,2,3.$$

Inspired by results given in [26], we have the following theorem

Theorem 3.3.3 *If the bifurcation parameter α crosses the critical value $\alpha = \alpha^* \in (0,1)$, then the fractional system (3.3), passes through a Hopf bifurcation at EE^*, if the given singularity conditions $(i),(ii)$ and also (iii) are satisfy where*

(i) The Jacobian matrix at EE^* equilibrium of the system (3.3) induces complex conjugate eigenvalues $\lambda_i = \varsigma(\alpha) + i\varsigma(\alpha)$, with $\alpha > 0$.

(ii) $H(\alpha^*) = 0$.

(iii) $\frac{dH(\alpha)}{d\alpha}|_{\alpha=\alpha^*} \neq 0$ (the transversality condition).

3.4 Global stability of the endemic equilibrium points

3.4.1 Global stability of disease-free-equilibrium

The global dynamical of the free steady state E^0 is established by the following theorem.

Theorem 3.4.1 *The disease-free-equilibrium E^0 of the system (3.3) globally asymptotically stable if $\mathcal{R}_0 \leq 1$.*

Proof 3.4.1 *If $\mathcal{R}_0 \leq 1$, it easy to verify that*

$$(\mu+\gamma)\beta_E(0)g'(0)h(S^0) + \beta_I(0)f'(0)h(S^0)\delta < (\mu+\gamma)(\mu+\delta),$$

but

$$(\mu+\gamma)\frac{\beta_E(0)g'(0)h(S^0)}{(\mu+\delta)} < (\mu+\gamma)\frac{\beta_E(0)g'(0)h(S^0)}{(\mu+\delta)} + \beta_I(0)f'(0)h(S^0),$$

that is

$$(\mu+\gamma)\beta_E(0)g'(0)h(S^0) < (\mu+\gamma)(\mu+\delta),$$

Now, we consider the following Lyapunov function :

$$V(E,I) = \delta E + bI,$$

where

$$b = \mu + \delta - \beta_E(0)g'(0)h(S^0),$$

it is easy to see that $b > 0$. According the fractional order derivative along of system (3.3) we get

$$D^{\alpha}V(E,I) = \delta D^{\alpha}E(t) + bD^{\alpha}I(t),$$

It follows from second and third equation of system (3.3) *that*

$$\begin{aligned} D^{\alpha}V(E,I) &= \delta\Big(\beta_E(E)g(E)h(S) + \beta_I(I)f(I)h(S) - (\mu+\delta)E\Big) \\ &\quad + b\Big(\delta E - (\mu+\gamma)I\Big), \\ &= \delta\beta_E(E)g(E)h(S) + \delta\beta_I(I)f(I)h(S) - (\mu+\delta)(\mu+\gamma)I \\ &\quad - \beta_E(0)g^{'}(0)h(S^0)\delta E + \beta_E(0)g^{'}(0)h(S^0)(\gamma+\mu)I, \end{aligned}$$

if (A_1) and (A_3) holds then $\beta_E(E) \le \beta_E(0)$, $\beta_I(I) \le \beta_I(0)$ and since $S \le S^0$, leads to

$$\begin{aligned} D^{\alpha}V(E,I) &\le \delta\beta_E(0)g(E)h(S^0) + \delta\beta_I(0)f(I)h(S^0) - (\mu+\delta)(\mu+\gamma)I \\ &\quad - \beta_E(0)g^{'}(0)h(S^0)\delta E + \beta_E(0)g^{'}(0)h(S^0)(\gamma+\mu)I, \end{aligned}$$

thus,

$$\begin{aligned} D^{\alpha}V(E,I) &\le \lim_{E\longrightarrow 0^+}\frac{g(E)}{E}\delta\beta_E(0)h(S^0)E + \lim_{I\longrightarrow 0^+}\frac{f(I)}{I}\delta\beta_I(0)h(S^0)I - (\mu+\delta)(\mu+\gamma)I \\ &\quad - \beta_E(0)g^{'}(0)h(S^0)\delta E + \beta_E(0)g^{'}(0)h(S^0)(\gamma+\mu)I, \\ &\le \Big(\delta\beta_I(0)f^{'}(0)h(S^0) - (\delta+\mu)(\gamma+\mu) + \beta_E(0)g^{'}(0)h(S^0)(\gamma+\mu)\Big)I \\ &\le (\delta+\mu)(\gamma+\mu)(\mathcal{R}_0 - 1)I, \end{aligned}$$

Furthermore, when $\mathcal{R}_0 < 1$ we have, $D^{\alpha}V(E,I) < 0$ and $D^{\alpha}V(E,I) = 0$ if and only if $I = 0$ replacing this last equation in the first and second equations of the right-hand side of (3.3) we find $E = 0$ and $S = S^0$, this implies that the largest invariant set where $D^{\alpha}V = 0$ is the singleton $\{E^0\}$. Hence, applying LaSalle's invariant principal [13], the solution of system (3.3) tend to E^0. Then the disease free equilibrium is globally asymptotically stable if $\mathcal{R}_0 <$ 1.

3.4.2 Global stability of the endemic equilibrium

In this subsection, we show that the endemic equilibrium EE^* is converges globally asymptotically stable, we recall that EE^* satisfies the system (3.4). Noting that for when $\mathcal{R}_0 > 1$ the disease-free equilibrium E^0 is unstable and endemic equilibrium EE^* is persistent in population. Then, in the following for the global stability of endemic equilibrium EE^*, we assume that $\mathcal{R}_0 > 1$ and the following inequality are holds

$$\left(\frac{\beta_E(E)g(E)h(S)}{\beta_E(E^*)g(E^*)h(S^*)} - 1\right)\left(\frac{S^*}{S} - \frac{\beta_E(E^*)g(E^*)h(S^*)}{\beta_E(E)g(E)h(S)}\right) < 0, \tag{3.7}$$

and

$$\left(\frac{\beta_I(I)f(I)h(S)}{\beta_I(I^*)f(I^*)h(S^*)} - 1\right)\left(\frac{S^*}{S} - \frac{\beta_I(I^*)f(I^*)h(S^*)}{\beta_I(I)f(I)h(S)}\right) < 0. \tag{3.8}$$

We have the following mains results.

Theorem 3.4.2 *Assume that $\mathcal{R}_0 > 1$. If inequalities (3.7) and (3.8) holds, then endemic equilibrium EE^* is globally asymptotically stable.*

Proof 3.4.2 *We consider the Lyapunov function*

$$V(S,E,I) = S^*H\Big(\frac{S}{S^*}\Big) + E^*H\Big(\frac{E}{E^*}\Big) + AI^*H\Big(\frac{I}{I^*}\Big),$$

where

$$A = \frac{\mu+\delta}{\delta},$$

and $H(x) = x - \ln(x) - 1$. Obviously, H is positive function and admits a minimun point at $x^* = 0$. Appliqueing the Caputo derivative of V along the trajectories of system (3.3) and using lemma 1.2.6, we find

$$D^\alpha V(S,E,I) \leq \Big(1 - \frac{S^*}{S}\Big)\frac{dS}{dt} + \Big(1 - \frac{E^*}{E}\Big)\frac{dE}{dt} + A\Big(1 - \frac{I^*}{I}\Big)\frac{dI}{dt}. \tag{3.9}$$

Using the first equation of system (3.3), we get

$$\Big(1 - \frac{S^*}{S}\Big)D^\alpha S \;=\; \Big(1 - \frac{S^*}{S}\Big)\Big(\lambda - \beta_E(E)g(E)h(S) - \beta_I(I)f(I)h(S) - \mu S\Big),$$

but

$$\mu S^* = \lambda - \beta_E(E^*)g(E^*)h(S^*) - \beta_I(I^*)f(I^*)h(S^*),$$

thus,

$$\begin{aligned}\Big(1-\frac{S^*}{S}\Big)D^\alpha S &= \lambda-\beta_E(E)g(E)h(S)-\beta_I(I)f(I)h(S)\\ &-\lambda\frac{S^*}{S}+\beta_E(E)g(E)h(S)\frac{S^*}{S}+\beta_I(I)f(I)h(S)\frac{S^*}{S}\\ &-\lambda\frac{S}{S^*}+\beta_E(E^*)g(E^*)h(S^*)\frac{S}{S^*}+\beta_I(I^*)f(I^*)h(S^*)\frac{S}{S^*}\\ &+\lambda-\beta_E(E^*)g(E^*)h(S^*)-\beta_I(I^*)f(I^*)h(S^*).\end{aligned}$$

Consequently,

$$\begin{aligned}\Big(1-\frac{S^*}{S}\Big)D^\alpha S &= \mu S^*(2-\frac{S^*}{S}-\frac{S}{S^*})+\beta_E(E^*)g(E^*)h(S^*)(-\frac{\beta_E(E)g(E)h(S)}{\beta_E(E^*)g(E^*)h(S^*)}\\ &+\frac{\beta_E(E)g(E)h(S)S^*}{\beta_E(E^*)g(E^*)h(S^*)S}+1-\frac{S^*}{S})+\beta_I(I^*)f(I^*)h(S^*)(-\frac{\beta_I(I)f(I)h(S)}{\beta_I(I^*)f(I^*)h(S^*)}\\ &+\frac{\beta_I(I)f(I)h(S)S^*}{\beta_E(I^*)f(I^*)h(S^*)S}+1-\frac{S^*}{S}).\end{aligned} \tag{3.10}$$

Now,

$$\Big(1-\frac{E^*}{E}\Big)D^\alpha E = \Big(1-\frac{E^*}{E}\Big)\Big(\beta_E(E)g(E)h(S)+\beta_I(I)f(I)h(S)-(\mu+\delta)E(t)\Big),$$

using the second equation of system (3.3), we get

$$\beta_E(E^*)g(E^*)h(S^*)+\beta_I(I^*)f(I^*)h(S^*)=(\mu+\delta)E^*,$$

thus,

$$\begin{aligned}\Big(1-\frac{E^*}{E}\Big)D^\alpha E &= \beta_E(E)g(E)h(S)+\beta_I(I)f(I)h(S)\\ &-\beta_E(E)g(E)h(S)\frac{E^*}{E}-\beta_I(I)f(I)h(S)\frac{E^*}{E}+\beta_E(E^*)g(E^*)h(S^*)\\ &+\beta_I(I^*)f(I^*)h(S^*)-\beta_E(E^*)g(E^*)h(S^*)\frac{E}{E^*}-\beta_I(I^*)f(I^*)h(S^*)\frac{E}{E^*},\end{aligned}$$

which leads to

$$\begin{aligned}\Big(1-\frac{E^*}{E}\Big)D^\alpha E &= \beta_E(E^*)g(E^*)h(S^*)(\frac{\beta_E(E)g(E)h(S)}{\beta_E(E^*)g(E^*)h(S^*)}-\frac{\beta_E(E)g(E)h(S)E^*}{\beta_E(E^*)g(E^*)h(S^*)E}+1-\frac{E}{E^*})\\ &+\beta_I(I^*)f(I^*)h(S^*)(\frac{\beta_I(I)f(I)h(S)}{\beta_I(I^*)f(I^*)h(S^*)}-\frac{\beta_I(I)f(I)h(S)E^*}{\beta_I(I^*)f(I^*)h(S^*)E}+1-\frac{E}{E^*}).\end{aligned} \tag{3.11}$$

Finally,

$$A\Big(1-\frac{I^*}{I}\Big)D^\alpha I = A\Big(1-\frac{I^*}{I}\Big)\Big(\delta E-(\mu+\gamma)I\Big),$$

using the third equation of system (3.3), we obtain

$$(\mu+\gamma)I^* = \delta E^* \ ,$$

it follows that,

$$\begin{aligned} A\Big(1-\frac{I^*}{I}\Big)D^\alpha I &= A(\delta E-(\mu+\gamma)I-\delta E\frac{I^*}{I}+(\mu+\gamma)I^*),\\ &= A(\delta E-(\mu+\gamma)I-\delta E\frac{I^*}{I}+\delta E^*),\\ &= \beta_E(E^*)g(E^*)h(S^*)(\frac{E}{E^*}-\frac{I}{I^*}-\frac{I^*E}{IE^*}+1),\\ &\quad +\beta_I(I^*)f(I^*)h(S^*)(\frac{E}{E^*}-\frac{I}{I^*}-\frac{I^*E}{IE^*}+1), \end{aligned}$$

then,

$$\begin{aligned} \Big(1-\frac{I^*}{I}\Big)\frac{dI}{dt} &= \beta_E(E^*)g(E^*)h(S^*)(\frac{E}{E^*}-\frac{I}{I^*}-\frac{I^*E}{IE^*}+1),\\ &\quad +\beta_I(I^*)f(I^*)h(S^*)(\frac{E}{E^*}-\frac{I}{I^*}-\frac{I^*E}{IE^*}+1). \end{aligned} \tag{4.12}$$

Summing the equation(3.10),(3.11) and (3.12), we get

$$\begin{aligned} D^\alpha V(S,E,I) &= \mu S^*(2-\frac{S^*}{S}-\frac{S}{S^*})+\beta_E(E^*)g(E^*)h(S^*)(3+\frac{\beta_E(E)g(E)h(S)S^*}{\beta_E(E^*)g(E^*)h(S^*)S}\\ &\quad -\frac{\beta_E(E)g(E)h(S)E^*}{\beta_E(E^*)g(E^*)h(S^*)E}-\frac{I}{I^*}-\frac{I^*E}{IE^*}-\frac{S^*}{S})+\beta_I(I^*)f(I^*)h(S^*)(3\\ &\quad +\frac{\beta_I(I)f(I)h(S)S^*}{\beta_I(I^*)f(I^*)h(S^*)S}-\frac{\beta_I(I)f(I)h(S)E^*}{\beta_I(I^*)f(I^*)h(S^*)E}-\frac{I}{I^*}-\frac{I^*E}{IE^*}-\frac{S^*}{S}), \end{aligned}$$

after some calculation we get

$$\begin{aligned} D^\alpha V(S,E,I) &= \mu S^*(2-\frac{S^*}{S}-\frac{S}{S^*})\\ &\quad +\beta_E(E^*)g(E^*)h(S^*)(4+(\frac{\beta_E(E)g(E)h(S)}{\beta_E(E^*)g(E^*)h(S^*)}-1)(\frac{S^*}{S}-\frac{\beta_E(E^*)g(E^*)h(S^*)}{\beta_E(E)g(E)h(S)})\\ &\quad -\frac{\beta_E(E)g(E)h(S)E^*}{\beta_E(E^*)g(E^*)h(S^*)E}-\frac{I}{I^*}-\frac{I^*E}{IE^*}-\frac{\beta_E(E^*)g(E^*)h(S^*)}{\beta_E(E)g(E)h(S)})\\ &\quad +\beta_I(I^*)f(I^*)h(S^*)(4+(\frac{\beta_I(I)f(I)h(S)}{\beta_I(I^*)f(I^*)h(S^*)}-1)(\frac{S^*}{S}-\frac{\beta_I(I^*)f(I^*)h(S^*)}{\beta_I(I)f(I)h(S)})\\ &\quad -\frac{\beta_I(I)f(I)h(S)E^*}{\beta_I(I^*)f(I^*)h(S^*)E}-\frac{I}{I^*}-\frac{I^*E}{IE^*}-\frac{\beta_I(I^*)f(I^*)h(S^*)}{\beta_I(I)f(I)h(S)}), \end{aligned}$$

using the arithmetic means we get

$$2-\frac{S^*}{S}-\frac{S}{S^*}<0,$$

if

$$\left(\frac{\beta_E(E)g(E)h(S)}{\beta_E(E^*)g(E^*)h(S^*)} - 1\right)\left(\frac{S^*}{S} - \frac{\beta_E(E^*)g(E^*)h(S^*)}{\beta_E(E)g(E)h(S)}\right) < 0,$$

and

$$\left(\frac{\beta_I(I)f(I)h(S)}{\beta_I(I^*)f(I^*)h(S^*)} - 1\right)\left(\frac{S^*}{S} - \frac{\beta_I(I^*)f(I^*)h(S^*)}{\beta_I(I)f(I)h(S)}\right) < 0,$$

we get

$$D^\alpha V \leq 0,$$

the equality $D^\alpha V = 0$ holds if and only if $S = S^*, E = E^*, I = I^*$. Therefore, By Lyapunov-LaSalle principle theorem [30], $\left\{(S^*, EE^*, I^*)\right\}$ is the largest compact invariant set in Γ this implies that (S^*, EE^*, I^*) is globally asymptotically stable.

3.5 Numerical simulations

In this section, we investigate to numerical solution of system (3.4), using the Adams-type predictor method [12]. We select the parameters values as : $\beta_I = 0.25, \beta_E = 0.03, \lambda = 0.06, \mu = 0.05, \gamma = 0.4, \delta = 0.3, \alpha = 0.95$, for initials conditions $(S_0, E_0, I_0, R_0) = (1 - 10^{-7}, 10^{-7}, 0, 0)^T$.

In table 2.1 by using the next-generation matrix method, we can compute the basic reproduction number for different incidence function, where $c_1 = 0.1, c_2 = 0.8, c_3 = 1$. The approximation solution are given is Figure 2.2.

We remark that the susceptible individuals tends to $S^0 = 1.2$ and for a long time, the others persons numbers converge to zero. According to theorem 3.4.1, the solution of our system converge global to the free-steady states $E^0 = (1.2, 0, 0, 0)^T$.

Table 3.1: The value of basic reproduction number for different incidence function when $\mathscr{R}_0 < 1$.

$f(I)$	$g(E)$	$h(S)$	R_0
I	E	S	0.6743
$\frac{I}{1+c_1 I}$	$\frac{E}{1+c_2 E}$	S	0.6743
I	E	$\frac{S^2}{1+c_3 S}$	0.4458

If we choose the parameters values as : $\beta_I = 0.5, \beta_E = 0.3, \lambda = 0.6, \mu = 0.1, \gamma = 0.2, \delta = 0.3, \alpha = 0.95$, for initials conditions $(S_0, E_0, I_0, R_0) = (1 - 10^{-7}, 10^{-7}, 0, 0)^T$. In

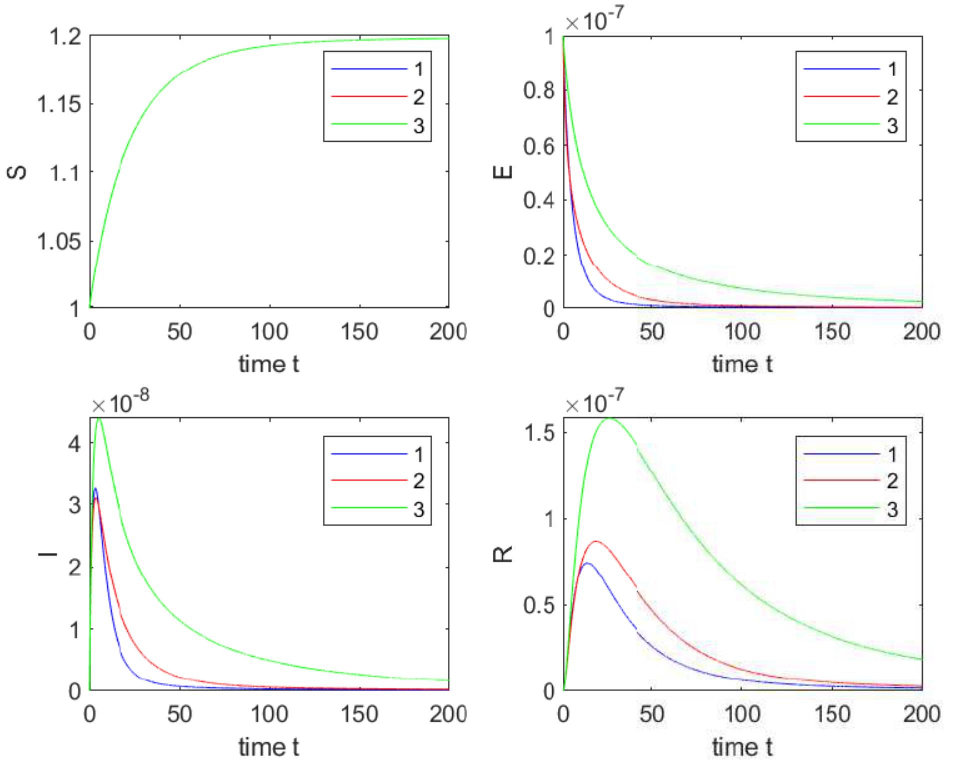


Figure 3.2: Dynamics of S, E, I and R where $\mathscr{R}_0 < 1$, the green line for $f(I) = \frac{I}{1+c_1 I}, g(E) = \frac{E}{1+c_2 E}, h(S) = S$, the blue line for $f(I) = \frac{S^2}{1+c_3 S}, g(E) = E, h(S) = S$, the red line for $f(I) = I, g(E) = E, h(S) = S$.

Table 3.2: The value of basic reproduction number for different incidence function when $\mathscr{R}_0 > 1$.

$f(I)$	$g(E)$	$h(S)$	$\mathscr{R}_0$
I	E	S	1.7576
$\frac{I}{1+c_1 I}$	$\frac{E}{1+c_2 E}$	S	1.7576
I	E	$\frac{S^2}{1+c_3 S}$	2.5108

table 2.2 by using the next-generation matrix method, we can compute the basic reproduction number for different incidence function, where $c_1 = 0.1, c_2 = 0.8, c_3 = 0.2$.
The Figure 2.3 shows that the solution converges to one endemic equilibrium from theorem 3.4.2 the equilibrium EE^* is globally asymptotically stable. If we compare between the behavior on the classes S, E, I and R for the two incidence function, obviously, we can see that the incidence function can not change the behavior of the solution and the peak of infected people but it can change the number of any compartment.

Now, we simulate the system (3.4) for different values of α. From figure 2.4, we can observe that the size of susceptible is increased when the parameter α increases, but we look the opposite for the other individuals, witch implies that our model depends to fractional order α. For his memory and non-locally effect, the fractional order parameter very important to modeling the spread of epidemic.

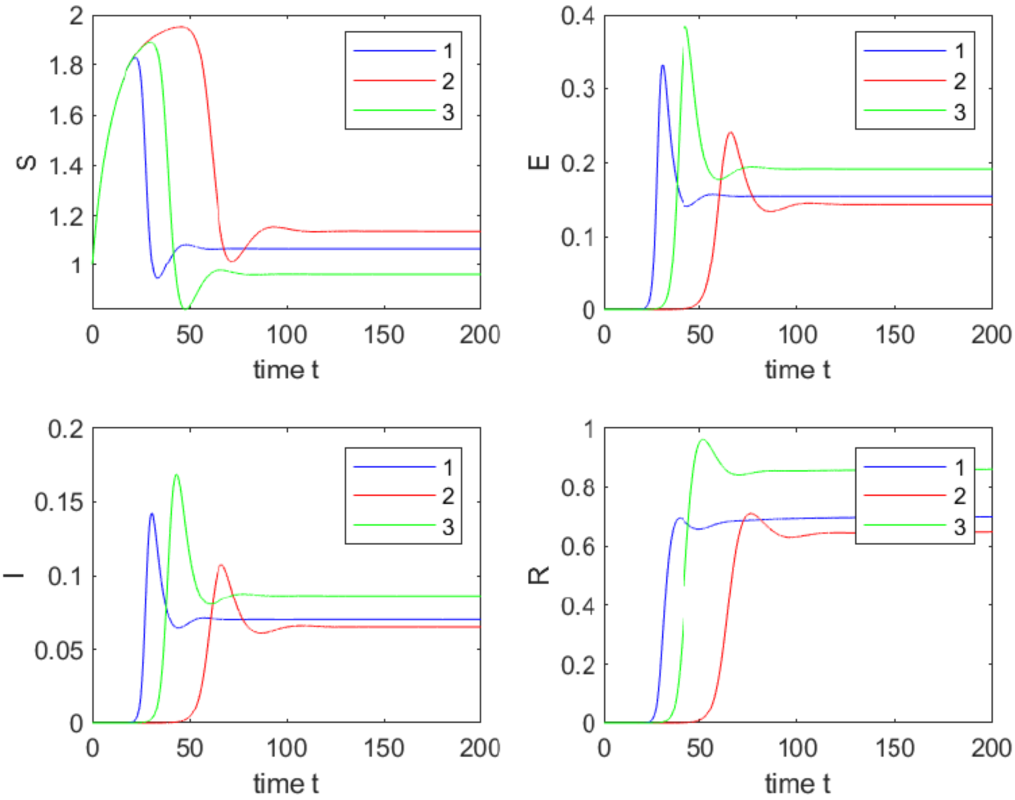


Figure 3.3: Dynamics of S, E, I and R when $\mathscr{R}_0 > 1$, the green line for $f(I) = \frac{I}{1+c_1 I}, g(E) = \frac{E}{1+c_2 E}, h(S) = S$, the blue line for $f(I) = \frac{S^2}{1+c_3 S}, g(E) = E, h(S) = S$, the red line for $f(I) = I, g(E) = E, h(S) = S$.

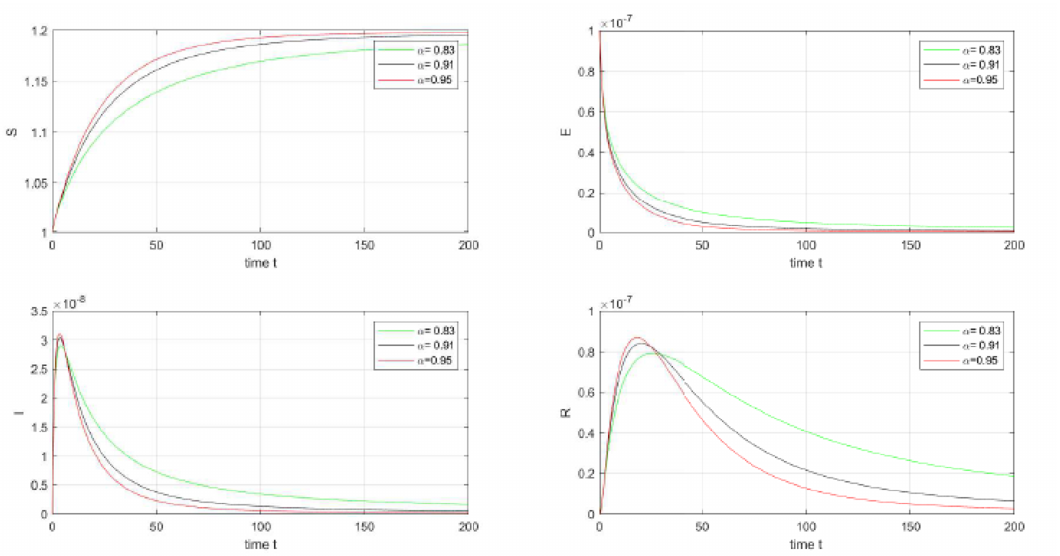


Figure 3.4: Dynamics of S, E, I and R for different values of α.

CONCLUSION AND PERSPECTIVES

In chapter 2, We have studied a fractional-order $SEIR$ epidemic model with time delay and general incidence rate function. The next generation matrix approach is used to calculate the basic reproduction number. We have obtained necessary and sufficient conditions for local stability of both free and endemic steady states. We have proved the global stability of the two steady states under some conditions on the incidence function using the method of Lyapunov function. We use this model to explain the Algerian COVID-19 pandemic at the onset, which occurred in late February 2020. Numerical simulations show that recovered populations grows quickly whereas the infected and exposed populations decrease significantly. That leaves us with the conclusion that most people will recover. Additionally, a sample of infected persons is seen around 160 days after the illness first manifests, indicating an incubation period of approximately $8 \sim 10$ days. The WHO figures show this peak, which corresponds to late July (see figure 8). In the end, numerical simulations demonstrate that the number of COVID-19 healthy persons will be maximized by using a fractional-order derivative. The delayed fractional-order model that we provide can more precisely describe the Algerian data at the start of the infection and prior to the lock and isolation processes than the reality without delay and with classic derivative.

In chapter 3, we have presented a general nonlinear fractional epidemic model in the sense of Caputo derivative. For controlling the epidemic model, we have calculated a threshold concept biological called the basic reproduction number. The local stability of both steady states is studied. By using the Lyapunov direct method, we prove the global stability of the healthy steady state when the number of reproduction is less than one, which means that the disease has died out. For numerical simulations, we have given three types of the incidence function. We obtain that the incidence function can not influence the behavior of steady states but can change the number of any compartment of the population. In addition, we obtain that the fractional order derivative is very important of epidemic modeling for its memory effect. The numerical simulation confirms the theory study.

We can then conclude that the introduction of time delay and fractional derivative in epidemic models can describe more really the spreading of an epidemic. The models we have proposed are a generalization of many other models in literature and can be used to describe many other types of epidemics.
From our perspective, we estimate this models for age structured populations and diffusion.

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