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Inference in periodic restricted EXPAR models

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Abstract

This thesis investigates the probabilistic and statistical properties of the periodic restricted exponential autoregressive (PEXPAR) process. By leveraging Markov chain theory, we establish conditions for strict periodic stationarity. Parameter estimation is performed using the quasi-maximum likelihood (QML) method, while model adequacy is assessed through Wald, Likelihood Ratio (LR) and Lagrange Multiplier (LM) tests to evaluate linearity and the nullity of final coefficients. Simulation studies support these findings by demonstrating the accuracy of the proposed tests in maintaining correct nominal levels and increasing power as the sample size grows, thereby validating the practical applicability of the methodology across various scenarios.

In addition, we propose a recursive estimation algorithm for the restricted exponential autoregressive (EXPAR) model. This algorithm, based on the matrix inversion lemma, allows for efficient online estimation through Recursive Least Squares (RLS). It is shown that the RLS estimators are asymptotically efficient. A short simulation study highlights the excellent performance of the proposed estimators. Finally, we apply both periodic autoregressive (PAR) and PEXPAR models to precipitation time series data from Algeria, illustrating the practical relevance of our findings.

Keywords and phrases: Periodic restricted EXPAR, strict periodic stationarity, QMLE, LR and LM tests, RLS and on line estimation algorithm,.

Résumé

Cette thèse étudie les propriétés probabilistes et statistiques du processus autorégressif exponentiel restreint périodique (PEXPAR). En s'appuyant sur la théorie des chaînes de Markov, nous établissons les conditions de stationnarité périodique stricte. L'estimation des paramètres est réalisée à l'aide de la méthode du quasi maximum de vraisemblance (QML), tandis que l'adéquation du modèle est évaluée par les tests du rapport de vraisemblance (LR) et du multiplicateur de Lagrange (LM) afin d'examiner la linéarité et la nullité des coefficients finaux. Des études de simulation viennent appuyer ces résultats en montrant la capacité des tests proposés à maintenir des niveaux nominaux corrects et une puissance croissante avec la taille de l'échantillon, ce qui valide l'applicabilité pratique de la méthodologie dans divers scénarios.

Par ailleurs, nous proposons un algorithme d'estimation récursif pour le modèle autorégressif exponentiel restreint (EXPAR). Cet algorithme, basé sur le lemme d'inversion matricielle, permet une estimation en ligne efficace via la méthode des moindres carrés récursifs (RLS). Nous montrons que les estimateurs RLS sont asymptotiquement efficaces. Une courte étude de simulation met en évidence l'excellente performance des estimateurs proposés. Enfin, nous appliquons les modèles autorégressif périodique (PAR) et PEXPAR à des séries temporelles de précipitations en Algérie, illustrant ainsi la pertinence pratique de nos résultats.

الملخص

تتناول هذه الاطروحة دراسة الخصائص الاحتمالية والإحصائية لنموذج الانحدار الذاتي الأسي المقيد الدوري .(PEXPAR).

بالاعتماد على نظرية سلاسل ماركوف، قمنا بتحديد شروط الاستقرارية الدورية الصارمة. ثم قدرنا المعاملات باستخدام طريقة الاحتمال الأعظمي شبه الكامل (QML)بينما يتم قييمنا ملاءمة النموذج

من خلال اختبارات المعلمات (Wald, LR, LM) بفحص الخطية وانعدام المعاملات النهائية.دعمت المحاكاة النتائج المحصل عليها، حيث تُظهر قدرة الاختبارات المقترحة على الحفاظ على المستويات الاسمية الصحيحة وزيادة القدرة .الإحصائية مع زيادة حجم العينة،،مما يؤكد صلاحية المنهجية وتطبيقها العملي في سيناريوهات مختلفة كما قمنا بتطبيق من الواقع على بيانات سلسلة نساقط الأمطار في الجزائر .

. بالإضافة إلى ذلك، نقترح خوارزمية تقدير تراجعية للنموذج الأسي المقيد (EXPAR) تعتمد على مبدأ معكوس المصفوفة مما يسمح بالتقدير الفوري والفعال باستخدام طريقة المربعات الصغرى التكرارية (RLS) نبرهن انها كفوءة تقريبًا مع زيادة العينة. كما تسلط دراسة محاكاة قصيرة الضوء على الأداء العالي للمقدرات R L S .

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Contents

Abbreviations

AR	Autoregressive model
EXPAR	Exponential Autoregressive
PEXPAR	Periodic Exponential Autoregressive
RLS	Recursive Least Squares
LS	Least Squares
QML	Quasi-Maximum Likelihood
LR	Likelihood Ratio
LM	Lagrange Multiplier
SPS	Strict Periodic Stationarity
PAR	Periodic Autoregressive
WN	White Noise

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Introduction

The Exponential Autoregressive (EXPAR) model, introduced by [21], stands as a cornerstone nonlinear-in-mean parametric model. It offers a flexible framework for capturing complex time series phenomena such as limit cycles, jump phenomena, and non-normality—features that often elude linear models. Despite its significance, accurately estimating the parameters of the EXPAR model remains a major challenge. The inherent nonlinearity of the model leads to a non-convex objective function, often riddled with multiple local minima, making the estimation of nonlinear coefficients particularly difficult. Various estimation techniques have been proposed to address this challenge. Early methods include Least Squares (LS) and Maximum Likelihood (ML), while more recent approaches, such as Quasi-ML (QML), Kalman filtering, genetic algorithms, and Bayesian techniques, have been developed to improve estimation precision. Notably, [24] and [25] introduced a heuristic method for real-time estimation of the nonlinear parameter directly from the data, which has proven indispensable in practical applications. For instance, [19] successfully applied this method to model vibrations during drilling. Simulation studies by [9] and [3] further revealed that the estimation of linear parameters stabilizes when the nonlinear parameter exceeds a threshold of 5. The EXPAR model's distinctive feature is its linearity in parameters while maintaining nonlinearity in variables, thus preserving all inherent nonlinear behaviors. From this model, the Restricted EXPAR model is derived, where the nonlinear parameter is treated as known—either from previous studies or estimated heuristically—thus facilitating simpler and more stable estimation procedures. Building on this foundation, [18] introduced the Periodic Restricted EXPAR (PEXPAR) model, extending the restricted framework to periodic time series characterized by seasonality in mean and autocovariance, while retaining the nonlinear structure of the original EXPAR model. This model is particularly suited for applications where periodic patterns are prominent, such as environmental and biomedical data. In this thesis, we aim to deepen the understanding of the PEXPAR model by studying its strict periodic stationarity and probabilistic properties. Our methodology is grounded in two theoretical pillars: the duality between periodic models and stationary vector processes, as established by [11], and the theory of Markov chains in discrete time and continuous state space, as developed by [20]. This theoretical framework, previously applied to the EXPAR model by [7], has been adapted to the periodic context by [1] and [4]. Additionally, we contribute to the methodological arsenal for estimating and testing parameters in the PEXPAR model. We adopt the Quasi-ML method for estimation and employ the Likelihood Ratio (LR) and Lagrange Multiplier (LM) tests for significance testing, as outlined in [14, 15]. Furthermore, recognizing the need for real-time applications, we propose a Recursive Least Squares (RLS) algorithm for the estimation of the Restricted EXPAR model. This algorithm leverages the matrix inversion lemma to avoid direct matrix inversion, resulting in an efficient and memory-friendly implementation. The recursive approach allows for continuous refinement of parameter estimates as new data become available, making it particularly valuable in fields like signal processing, control, and real-time monitoring. By integrating theoretical insights with practical algorithms, this work aims to advance the modeling, estimation, and application of nonlinear periodic time series models, offering robust tools for analyzing complex real-world data. This thesis is organized into three chapters.

Chapter 1 provides the necessary preliminaries, where we recall the fundamental concepts and mathematical tools that will be used throughout the work.

Chapter 2 is devoted to inference for the Restricted PEXPAR(p) model. In this chapter, we establish the stationarity condition, study the estimation procedures, and develop statistical tests for the model coefficients. The theoretical results are complemented by simulation experiments and a real data application.

Chapter 3 focuses on recursive estimation in Restricted Exponential Autoregressive models. We analyze the asymptotic properties of the recursive algorithms and assess their performance through simulation studies.

Overall, the thesis aims to provide a rigorous statistical framework for Restricted Exponential Autoregressive models, combining theoretical results with practical tools for inference and estimation.

Chapter 1

Preliminaries

1.1 Time series models and their properties

1.1.1 Stationarity

Stationarity is a fundamental concept in time series analysis, as it serves as a natural generalization of the independent and identically distributed (i.i.d.) assumption in standard statistics. While i.i.d. observations are rarely encountered in real-world time series data, the assumption of stationarity provides a flexible and meaningful framework for analyzing and modeling time-dependent structures.

Definition 1 (Strict Stationarity)**.** *A time series $\{X_t,\, t \in \mathbb{Z}\}$ is said to be* ***strictly stationary*** *if the joint distributions of*

$$(X_{t_1}, \ldots, X_{t_k})' \quad \textit{and} \quad (X_{t_1+h}, \ldots, X_{t_k+h})'$$

are identical, for any positive integer k and any time indices $t_1, \ldots, t_k, h \in \mathbb{Z}$.

Definition 2 (Second-Order Stationarity)**.** *A time series $\{X_t,\, t \in \mathbb{Z}\}$ is said to be* ***second-order stationary*** *(or weakly stationary) if:*

(i) $\mathbb{E}[X_t^2] < \infty$ *for all* $t \in \mathbb{Z}$*,*

(ii) $\mathbb{E}[X_t] = m$ *for all* $t \in \mathbb{Z}$*,*

(iii) $\mathrm{Cov}(X_t, X_{t+h}) = \gamma_X(h)$ *depends only on the lag* h*.*

The simplest example of a second-order stationary process is white noise, which is particularly important as a building block for constructing more complex time series models.

Definition 3 (White Noise)**.** *A time series* $\{\varepsilon_t\}$ *is called a (weak)* ***white noise process*** *if, for some constant* σ^2*,*

(i) $\mathbb{E}[\varepsilon_t] = 0$ *for all* $t \in \mathbb{Z}$*,*

(ii)

$$\mathrm{Cov}(\varepsilon_t, \varepsilon_{t+h}) = \begin{cases} 0 & \text{if } h \neq 0, \\ \sigma^2 < \infty & \text{if } h = 0. \end{cases}$$

1.1.2 Periodic Stationarity

In the wide sense, periodicity in time series refers to regular patterns or fluctuations that repeat over a given time span. We denote the period of a time series by S; for instance, quarterly and monthly series are periodic time series with periods $S = 4$ and $S = 12$, respectively.

Let $\{X_t\}$ be a time series with period S. *Gladyshev (1961)* introduced the following definition:

Definition 4 (Periodically Stationary)**.** *A time series* $\{X_t\}$ *is said to be* ***periodically stationary*** *of period* S *if, for all integers* s, t*,*

(i) $\mathbb{E}[X_t^2] < \infty$*,*

(ii) $\mathbb{E}[X_t] = \mathbb{E}[X_{t+S}]$*,*

(iii) $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+S}, X_{t+S})$*.*

The last two properties imply that the mean and covariance functions of a periodically stationary time series are S-periodic in t. It is important to note that a periodically stationary process is not strictly stationary in the classical sense.

1.1.3 Advanced Concepts in Periodic Stationarity

Strict Periodic Stationarity (S.P.S)

A process X_t is said to be **strictly periodically stationary** with period S if

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_k})' = (X_{t_1+Sh}, X_{t_2+Sh}, \ldots, X_{t_k+Sh})',$$

for any positive integer k, time indices $t_1, \ldots, t_k$, and shift $h \in \mathbb{Z}$.

Periodic Ergodicity

A process X_t is said to be **periodically ergodic** if, for any Borel set B and integer m,

$$\frac{1}{n}\sum_{t=1}^{n} I_B\big(X_{s+St}, X_{s+1+St}, \ldots, X_{s+m+St}\big) \longrightarrow \mathbb{P}\big((X_s, X_{s+1}, \ldots, X_{s+m}) \in B\big), \quad \forall\, 1 \le s \le S,$$

as $n \to \infty$, where $I_B(\cdot)$ is the indicator function.

Gladyshev's Representation

A periodic time series X_t with period S can be represented as an S-variate non-periodic process $\underline{X}_\tau$, where

$$\underline{X}_\tau = \big(X_{1+S\tau}, \ldots, X_{S+S\tau}\big)'.$$

Consequence

X_t is strictly periodically stationary (or periodically ergodic) if and only if $\underline{X}_\tau$ is strictly stationary (or ergodic).

1.1.4 Importance of Periodic Stationarity in Time Series Analysis

Periodic stationarity refers to a stochastic process whose statistical properties (mean, variance, autocovariance, etc.) vary periodically over time. Unlike strictly stationary processes, where these properties remain constant, periodic stationarity allows for recurring fluctuations over a fixed interval.

Many real-world time series, such as those in climate science, economics, and hydrology, exhibit regular cycles that cannot be captured by classical stationary models. For example, temperature, rainfall, and economic indicators often show seasonal effects that recur annually or quarterly. Traditional stationary models fail to account for such cyclic variations, leading to poor fit and unreliable forecasts in the presence of periodicity.

Periodic stationarity allows the modeling of these cyclical patterns by capturing recurring changes in means, variances, and autocovariances. This leads to a more accurate understanding and improved forecasting of time series exhibiting seasonality or other regular cycles.

1.1.5 Examples of Periodically Stationary Processes

We present a few illustrative examples of processes that satisfy the definition of periodic stationarity.

Example 1: Periodic Mean Model

Consider the process

$$X_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim WN(0, \sigma^2),$$

where the mean function satisfies $\mu_{t+S} = \mu_t$. Since the mean is S-periodic and the innovations are white noise, X_t is a periodically stationary process of period S.

Example 2: Periodic Autoregressive Model (PAR(1))

A simple periodically autoregressive model of order one is given by

$$X_t = \phi_t X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN(0, \sigma^2),$$

where the autoregressive coefficient is periodic with period S, i.e.,

$$\phi_{t+S} = \phi_t.$$

In this case, both the mean and the autocovariance functions of X_t are periodic in t, which makes X_t a periodically stationary process.

Example 3: Seasonal White Noise

Let

$$X_t = \varepsilon_t, \qquad \varepsilon_t \sim WN(0, \sigma_t^2),$$

with $\sigma_{t+S}^2 = \sigma_t^2$. Here, the variance itself is periodic, while the mean is zero. This process is not stationary in the classical sense but is periodically stationary.

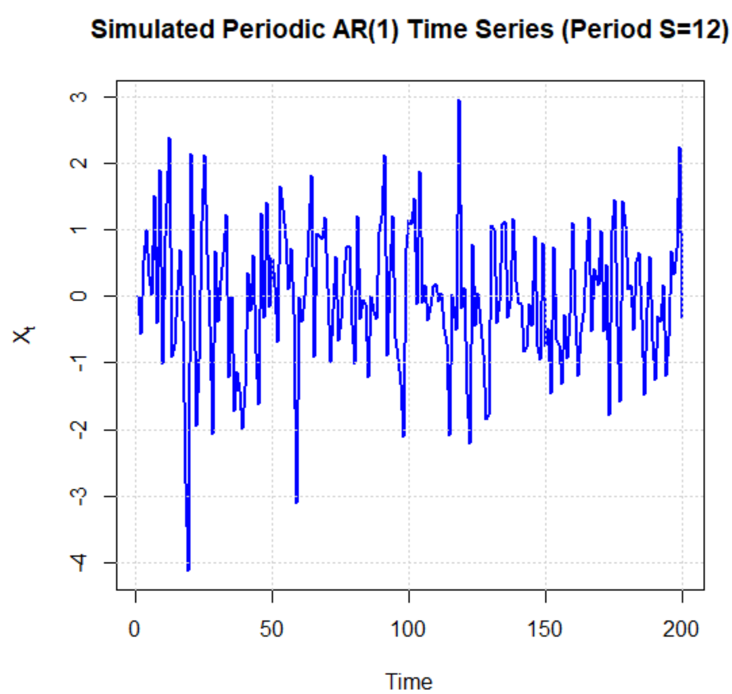


Figure 1.1: Simulated Periodic Time Series.

1.1.6 Reminder on the Stability of Continuous-State Markov Chains

We recall some fundamental results on the stability of Markov chains with continuous state space (see [20].

Let $(X_t)_{t\geq 0}$ be a Markov chain on a measurable state space $(E, \mathcal{E})$ with transition kernel $P(x, B)$, where for each $x \in E$, $P(x, \cdot)$ is a probability measure on $(E, \mathcal{E})$. For a probability measure π on E, we say that π is an *invariant measure* if

$$\pi(B) = \int_E P(x, B)\, \pi(dx), \quad \forall B \in \mathcal{E}.$$

Basic notions.

- **Irreducibility:** The chain is ψ-irreducible if for any measurable set B with $\psi(B) > 0$ and any $x \in E$, there exists $t \geq 1$ such that $P^t(x, B) > 0$.
- **Aperiodicity:** Defined in terms of "small sets"; it ensures that the chain does not exhibit cyclic behavior.
- **Recurrence:** The chain is recurrent if it returns infinitely often to sets of positive measure. It is *positive recurrent* if the mean return time is finite.
- **Stationarity:** Positive recurrence ensures the existence of an invariant probability measure π, which is unique under irreducibility.

Feller chains. A Markov chain is called a *Feller chain* if its transition operator P maps the space $C_b(E)$ of bounded continuous functions into itself, i.e.,

$$Pf(x) = \int_E f(y)\, P(x, dy) \in C_b(E), \quad \forall f \in C_b(E).$$

Geometric ergodicity. The chain is said to be geometrically ergodic if there exists $\rho > 1$ such that

$$\lim_{t\to\infty} \rho^t \, \|P^t(x, \cdot) - \pi\|_{\mathrm{TV}} = 0, \quad \forall x \in E,$$

where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm.

Feigin and Tweedie Theorem (1985)

Studying geometric ergodicity of a continuous state space Markov chain can be difficult by directly checking recurrence, existence of an invariant distribution, and aperiodicity. Feigin and Tweedie (1985) provide a criterion based on regularity and drift conditions.

Theorem 1 (Feigin and Tweedie, 1985)**.** *Let (X_t) be a Markov chain on a state space $E \subseteq \mathbb{R}^d$ with transition kernel P. Assume that:*

1. *(X_t) is a **Feller chain**, i.e. for every bounded continuous function f, the function*
$$x \mapsto (Pf)(x) = \int_E f(y)P(x,dy)$$
is continuous;

2. *(X_t) is φ-irreducible for a non-trivial measure φ;*

3. *there exists a compact set $A \subset E$, a continuous function $V : E \to [1,\infty)$, and a constant $\delta > 0$ such that:*
   - *$\varphi(A) > 0$;*
   - *$V(x) \geq 1$ for all $x \in A$;*
   - *for all $x \notin A$:*
$$(PV)(x) = \int_E V(y)\, P(x,dy) \leq (1-\delta)\, V(x).$$

*Then the chain (X_t) is **geometrically ergodic**, i.e. it admits a unique invariant distribution π, and there exist $C(x) < \infty$ and $\rho \in (0,1)$ such that*
$$\|P^t(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq C(x)\, \rho^t, \quad \forall x \in E.$$

Remark. The drift condition means that the chain is "pulled back" towards a compact set A: outside A, the expected value of $V(X_{t+1})$ decreases proportionally to $V(x)$. The set A plays the role of a *small set* ensuring irreducibility and aperiodicity. These properties guarantee not only the existence of an invariant distribution, but also geometric convergence towards it.

1.2 Hypothesis Tests

1.2.1 Definitions and Intuitions

In hypothesis testing we assess whether the observed data provide evidence for or against a specific claim about the data–generating mechanism.

Definition 5 (Null and Alternative Hypotheses)**.** *Let Θ be the parameter space and let $X = (X_1, \ldots, X_n)$ denote the sample. A hypothesis test concerns*

$$H_0: \ \theta \in \Theta_0 \qquad vs. \qquad H_1: \ \theta \in \Theta_1 = \Theta \setminus \Theta_0.$$

A (non-randomized) test is specified by a critical region $C \subset \mathcal{X}^n$*; we* reject H_0 *when* $X \in C$ *and* do not reject *otherwise.*

Definition 6 (Type I and Type II Errors)**.**

- *A **Type I error** occurs when H_0 is true but the test rejects it: $X \in C$ with $\theta \in \Theta_0$ ("false positive").*
- *A **Type II error** occurs when H_1 is true but the test fails to reject H_0: $X \notin C$ with $\theta \in \Theta_1$ ("false negative").*

Definition 7 (Size, Power, and Power Function)**.** *The **size** (or significance level) of a test is*

$$\alpha \ = \ \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(X \in C).$$

*For $\theta \in \Theta_1$, the **power** is the probability of correctly rejecting H_0:*

$$1 - \beta(\theta) \ = \ \mathbb{P}_\theta(X \in C).$$

*The mapping $\pi(\theta) = \mathbb{P}_\theta(X \in C)$ is called the **power function**; on Θ_0 it equals the test's size (at most α), and on Θ_1 it measures the test's ability to detect deviations from H_0.*

Remark 1. *In practice one fixes a nominal level $\alpha \in (0, 1)$ (e.g., 0.05) and constructs a rejection region C such that $\sup_{\theta \in \Theta_0} \mathbb{P}_\theta(X \in C) \leq \alpha$. Larger power on Θ_1 is desirable.*

Table 1.1: Decisions vs. truth in hypothesis testing

	Truth: H_0 true	**Truth: H_1 true**
Decision: Reject H_0	Type I error (prob. $\leq \alpha$)	Correct (power $= 1 - \beta$)
Decision: Do not reject H_0	Correct	Type II error (prob. β)

Definition 8 (Test Statistic and Critical Value Formulation)**.** *Many tests are based on a statistic $T = T(X)$ and a rejection rule of the form*

$$\textit{Reject } H_0 \quad \Longleftrightarrow \quad T(X) \in \mathcal{C}_\alpha,$$

where $\mathcal{C}_\alpha$ is chosen so that $\mathbb{P}_\theta(T(X) \in \mathcal{C}_\alpha) \leq \alpha$ for all $\theta \in \Theta_0$.

1.2.2 Classical Hypothesis Tests: Wald, Likelihood Ratio, and Lagrange Multiplier

In this subsection, primarily based on [8], we present and interpret the basic forms of three classical hypothesis tests: the Wald test, the Likelihood Ratio (LR) test, and the Lagrange Multiplier (LM) test. We assume that the likelihood function satisfies the standard regularity conditions that allow for a second-order Taylor series expansion and the interchange of integration and differentiation. Furthermore, it is assumed that the Fisher information matrix is non-singular, ensuring that parameters are (locally) identified.

Setup

Consider a simple hypothesis testing problem where the data

$$x = (x_0, x_1, \ldots, x_n)$$

are generated under the null hypothesis by a joint density $f(x, \theta^0)$, and under the alternative hypothesis by $f(x, \theta)$ with $\theta \in \mathbb{R}^k$. This corresponds to a test of a *simple null hypothesis* against a *composite alternative*.

Definition 9 (Log-likelihood Function). *The log-likelihood function is defined as*

$$L(\theta, x) = \log f(x, \theta).$$

It is maximized at $\hat{\theta}$ satisfying the first-order condition

$$\frac{\partial L}{\partial \theta}(\hat{\theta}, x) = 0.$$

Definition 10 (Score Function). *The score function is the gradient of the log-likelihood:*

$$s(\theta, x) = \frac{\partial L(\theta, x)}{\partial \theta}.$$

The maximum likelihood estimator (MLE) $\hat{\theta}$ is characterized by $s(\hat{\theta}, x) = 0$.

Definition 11 (Fisher Information). *The Fisher information matrix is defined as*

$$\mathcal{I}(\theta) = -\frac{1}{n}\,\mathbb{E}\left[\frac{\partial^2 L(\theta, x)}{\partial \theta\, \partial \theta'}\right].$$

Under regularity conditions, the asymptotic covariance of $\hat{\theta}$ is given by

$$W(\theta) = \frac{1}{n}\,\mathcal{I}(\theta)^{-1}.$$

The Wald Test

The Wald test is based on the difference between the unrestricted estimate $\hat{\theta}$ and the hypothesized value θ^0. Assuming $\hat{\theta}$ is asymptotically normal and $\mathcal{I}(\theta)$ is consistently estimated, the Wald statistic is defined as

$$\xi_W = n(\hat{\theta} - \theta^0)'\mathcal{I}(\theta)(\hat{\theta} - \theta^0),$$

which has an asymptotic $\chi^2(k)$ distribution with k degrees of freedom under the null hypothesis. This test generalizes the familiar t and F tests commonly used in econometrics.

The Likelihood Ratio Test

The Likelihood Ratio (LR) test compares the maximum value of the log-likelihood under the null hypothesis and the alternative hypothesis. Specifically, the test statistic is given by

$$\xi_{LR} = -2\big(L(\theta^0, x) - L(\hat{\theta}, x)\big),$$

which follows an asymptotic $\chi^2(k)$ distribution under the null hypothesis. This fundamental result was first derived in general form by [26].

The Lagrange Multiplier (Score) Test

The Lagrange Multiplier (LM) test, also known as the score test, is derived from the principle of constrained maximization. Specifically, we maximize the log-likelihood function subject to the constraint $\theta = \theta^0$. The associated Lagrangian is

$$H = L(\theta, x) - \lambda'(\theta - \theta^0),$$

with first-order conditions

$$\frac{\partial L}{\partial \theta}(\theta, x) = \lambda, \quad \theta = \theta^0.$$

Thus, the Lagrange multiplier λ equals the score evaluated at θ^0, i.e., $\lambda = s(\theta^0, x)$. The LM test statistic is

$$\xi_{LM} = \frac{1}{n} s(\theta^0, x)' \mathcal{I}^{-1}(\theta^0) s(\theta^0, x),$$

which again follows an asymptotic $\chi^2(k)$ distribution under the null hypothesis. This test was originally proposed by [23].

Geometric Interpretation

Each of these three tests provides a different perspective on measuring the discrepancy between the null and the alternative hypotheses:

- The Wald test compares the parameter estimates $\hat{\theta}$ and θ^0 (*horizontal distance*).
- The Likelihood Ratio test compares the log-likelihood values at $\hat{\theta}$ and θ^0 (*vertical distance*).
- The Lagrange Multiplier test examines the slope of the log-likelihood at θ^0 (*the gradient*).

The Wald test focuses on the horizontal difference, the LR test on the vertical difference, and the LM test on the slope at the null value.

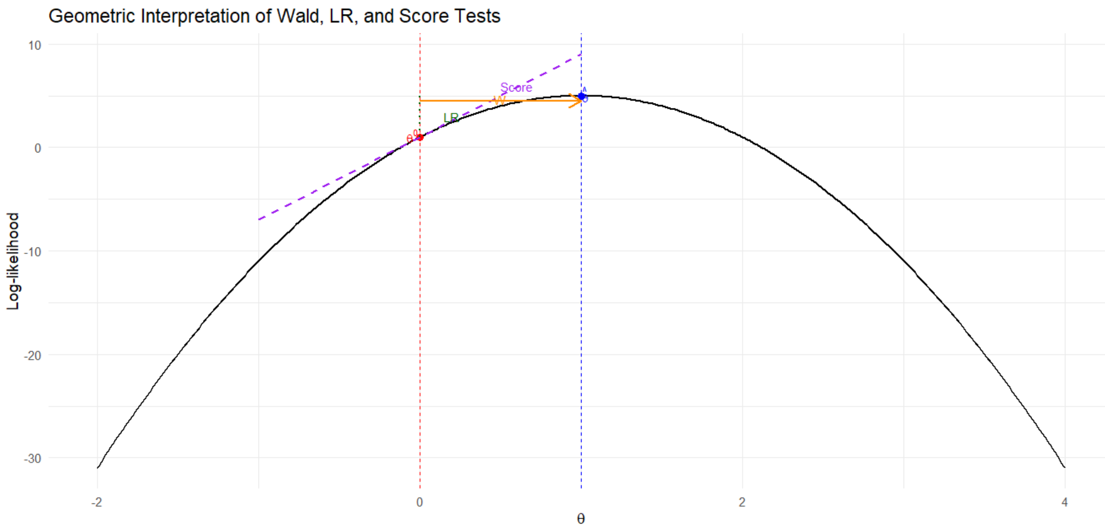


Figure 1.2: Geometric Interpretation.

Chapter 2

Inference of the Restricted PEXPAR(p) Model

2.1 Introduction

The Exponential Autoregressive model which was introduced by OZAki in 1980 , stands as a pivotal nonlinear-in-mean parametric model. It provides a flexible framework capable of capturing complex time series phenomena such as limit cycles, jump phenomena, and non-normality, which often elude linear models. Despite its significance, the accurate estimation of EXPAR model parameters remains a challenge.

Various estimation approaches have been explored, ranging from Least Squares (LS) and Maximum Likelihood (ML) to more recent methods such as Quasi Maximum Likelihood (QML), Kalman filtering, genetic algorithms, and Bayesian analysis, all aimed at enhancing estimation precision. Recent studies [6, 27, 28] underscore the growing interest in refining estimation accuracy. In particular, for real-time estimation scenarios, [24, 25] introduced a heuristic approach for estimating the nonlinear parameter directly from the data. This method is crucial due to the non-convex nature of the objective function governing the estimation of nonlinear coefficients, which often results in the presence of multiple local minima. Thus, the adoption of heuristic techniques becomes indispensable for effectively navigating these complexities and yielding practical solutions. Applications of these techniques, such as those by [19] in modeling vibrations during drilling, highlight their practical significance.

By leveraging such heuristic methods, one can then derive the Restricted EXPAR model. By contrast, [9, 3] observed through simulation studies that the estimation outcomes of the linear parameters exhibit close proximity when the nonlinear parameter exceeds a value of 5. The EXPAR model assumes linearity in its parameters while maintaining nonlinearity in the variables, thereby preserving all inherent nonlinear behaviors.

In this thesis, we focus on the periodic version of the restricted EXPAR model, termed the *Periodic Restricted EXPAR* (PEXPAR) model, introduced by [18]. This model caters to seasonal time series characterized by periodic mean and autocovariance, while retaining the nonlinear features observed in the original EXPAR framework. By investigating the strict periodic stationarity of this model, we aim to deepen our understanding of its probabilistic properties and provide methodological advancements in its estimation.

To accomplish our objective, we use two fundamental tools. First, we leveraged the duality between periodic models and stationary vector processes, as outlined in the seminal work of [11]. This duality allows us to seamlessly translate the periodic model into a vector process, thereby facilitating a deeper understanding of its underlying dynamics. Second, we utilized Markov chain theory in discrete time and continuous state space, as developed by [20]. This theoretical framework, previously applied to the EXPAR model by [7] and references therein, provides a robust analytical foundation for our investigation. Notably, this approach was tailored to the periodic context, drawing inspiration from prior studies such as [1, 4]. Through the integration of these methodologies and insights from the literature, we aim to enhance our understanding of the probabilistic properties of the periodic restricted EXPAR model.

Finally, through rigorous statistical analysis employing the Quasi Maximum Likelihood method for parameter estimation and Likelihood Ratio and Lagrange Multiplier tests for significance assessment, we contribute to the methodological arsenal for analyzing and modeling complex time series data. A comprehensive framework encompassing both testing and maximum likelihood estimation methodologies is provided by [15].

2.2 Strict Periodic Stationarity of the Restricted PEXPAR(p) Process

The Periodic Restricted Exponential AutoRegressive (PEXPAR$_S(p)$) process, with period S ($S \geq 2$), is given by

$$Z_t = \sum_{j=1}^{p} \left(\phi_{t,j} + \pi_{t,j}\ exp(-\gamma Z_{t-1}^2)\right) Z_{t-j} + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $\{\varepsilon_t; t \in \mathbb{Z}\}$ is a centered, independent and periodically distributed process with finite variance σ_t^2 and an unspecified probability density $f(\cdot)$. All the parameters and the innovation variance are periodic in time with period S. The slope parameter $\gamma > 0$ is assumed to be known. A heuristic determination from data is given by

$$\hat{\gamma} = -\frac{\log \varepsilon}{\max_t Z_t^2},$$

where ε is a small number (cf. [25]).

The Markov chain theory, applied in discrete time and continuous state space, serves as a valuable tool for analyzing the stationary properties of nonlinear time series. To address the classical stationary process theory, it is necessary to transform the periodic process into a non-periodic vector process, as outlined by [11]. In order to establish a Markovian representation, we define the lag vectors

$$\underline{Z}_t = (Z_t, Z_{t-1}, \ldots, Z_{t-p+1})' \in \mathbb{R}^p, \quad \underline{\varepsilon}_t = (\varepsilon_t, 0, \ldots, 0)' \in \mathbb{R}^p.$$

Then we have

$$\underline{Z}_t = \underline{A}_t \underline{Z}_{t-1} + \underline{\varepsilon}_t, \tag{2.2}$$

where

$$\underline{A}_t = \begin{pmatrix} \phi_{t,1} + \pi_{t,1} e^{-\gamma Z_{t-1}^2} & \cdots & \phi_{t,p} + \pi_{t,p} e^{-\gamma Z_{t-1}^2} \\ I_{(p-1)\times(p-1)} & 0_{(p-1)\times 1} & \end{pmatrix},$$

and $\{\underline{\varepsilon}_t\}$ is an i.i.d. sequence of random vectors independent of $\underline{Z}_0$.

The S-periodically stationary process $\{\underline{Z}_t\}$ of dimension p is equivalent to an Sp-dimensional stationary vector process X_τ, where

$$X_\tau = (\underline{Z}_{1+S\tau}, \ldots, \underline{Z}_{S+S\tau})' \in \mathbb{R}^{pS}, \quad \epsilon_\tau = (\underline{\varepsilon}_{1+S\tau}, \ldots, \underline{\varepsilon}_{S+S\tau})' \in \mathbb{R}^{pS}.$$

Thus,

$$X_\tau = F_\tau X_{\tau-1} + H_\tau \epsilon_\tau, \tag{2.3}$$

where

$$F_\tau = \begin{pmatrix} 0 & \cdots & 0 & \underline{A}_{1+S\tau} \\ 0 & \cdots & 0 & \underline{A}_{2+S\tau}\underline{A}_{1+S\tau} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \prod_{i=1}^{S} \underline{A}_{i+S\tau} \end{pmatrix},$$

and

$$H_\tau = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \underline{A}_{2+S\tau} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \prod_{i=1}^{S-1} \underline{A}_{i+S\tau} & \prod_{i=1}^{S-2} \underline{A}_{i+S\tau} & \cdots & 1 \end{pmatrix}.$$

Proving that process (2.2) is strictly periodically stationary is equivalent to proving that process (2.3) is strictly stationary. The space $\mathbb{R}^{pS}$ is equipped with a norm, and λ denotes the Lebesgue measure on $(\mathbb{R}^{pS}, \mathcal{B}(\mathbb{R}^{pS}))$. We set $\rho(A)$ for the spectral radius of matrix A. The Markovian process of pS-dimensional observations $\{X_\tau\}$ is governed by its initial value X_0 and equation (2.3).

To present the theorem giving the condition of stationarity, we make the following assumptions and notations.

Assumption 1. *The law of ϵ_τ is absolutely continuous with respect to the Lebesgue measure λ and $\mathbb{E}(\|\epsilon_\tau\|) < \infty$.*

Denote by $\|\cdot\|$ the Euclidean norm and $|f|_\infty = \sup\{f(x), x \in \mathcal{X}\}$. For $j = 1, \ldots, p$, we have

$$|\phi_{t,j} + \pi_{t,j} e^{-\gamma Z_{t-1}^2}|_\infty = \max(|\phi_{i,j}|, |\phi_{i,j} + \pi_{i,j}|) = c_{t,j} < \infty.$$

Set

$$C_t = \begin{pmatrix} c_{t,1} & \cdots & c_{t,p} \\ I_{(p-1)\times(p-1)} & 0_{(p-1)\times 1} & \end{pmatrix},$$

which is periodic of period S inherited from $\underline{A}_t$, and define

$$\underline{F} = \begin{pmatrix} 0 & \cdots & 0 & C_1 \\ 0 & \cdots & 0 & C_2 C_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \prod_{i=1}^{S} C_i \end{pmatrix}, \quad \underline{H} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ C_2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \prod_{i=1}^{S-1} C_i & \prod_{i=1}^{S-2} C_i & \cdots & 1 \end{pmatrix}.$$

Theorem 2. *Under Assumption 1, if*

$$\rho(\underline{F}) < 1, \tag{2.4}$$

then the restricted PEXPAR$_S(p)$ model is strictly periodically stationary.

Proof. The technical proof relies on Theorem 3.2 in [10], which provides the geometric ergodicity criterion, and [7], which presents the strict stationarity condition of the EXPAR(p) model. We verify the three conditions of the geometric ergodicity criterion.

Firstly, the vector process X_τ constitutes a first-order Markov chain on $(\mathbb{R}^{pS}, \mathcal{B}(\mathbb{R}^{pS}))$. The transition probability is well-defined by its values on the rectangles. Let $B \in \mathcal{B}(\mathbb{R}^{pS})$, then

$$P(x, B) = \mathbb{P}(X_1 \in B \mid X_0 = \boldsymbol{x}) = \mathbb{P}(F_1 \boldsymbol{x} + H_1 \epsilon_1 \in B).$$

For the Feller property, we have

$$\mathbb{E}(g(X_\tau) \mid X_{\tau-1} = \boldsymbol{x}) = \mathbb{E}\big(g(F_\tau \boldsymbol{x} + H_\tau \epsilon_\tau)\big).$$

If g is continuous and bounded, then the function $\boldsymbol{x} \mapsto g(F_\tau \boldsymbol{x} + H_\tau Z)$ is continuous and bounded for all Z. By the dominated convergence theorem, it follows that $\mathbb{E}(g(X_\tau) \mid X_{\tau-1} = \boldsymbol{x})$ is continuous, so X_τ is a Feller chain.

The irreducibility and the aperiodicity follow from the fact that ε_τ is absolutely continuous with respect to the Lebesgue measure λ.

A natural choice of the test function is $W(\boldsymbol{x}) = 1+\|\boldsymbol{x}\|$, and if condition (2.4) is satisfied, then there exists $c < 1$, and $\tau \in \mathbb{N}^*$ such that $\|\underline{F}^\tau\| \leq c$. Then

$$\begin{aligned}
\mathbb{E}(W(X_\tau) \mid X_0 = \boldsymbol{x}) &= 1 + \mathbb{E}(\|X_\tau\| \mid X_0 = \boldsymbol{x}) \\
&= 1 + \mathbb{E}\left\|\prod_{i=1}^{\tau} F_i \boldsymbol{x} + \sum_{i=1}^{\tau} \prod_{j=i+1}^{\tau} F_j H_i \epsilon_i\right\| \\
&\leq 1 + \|\underline{F}^\tau \boldsymbol{x}\| + \mathbb{E}\left(\sum_{i=1}^{\tau} \underline{F}^{\tau-i} \underline{H} |\epsilon_i|\right).
\end{aligned}$$

Let $K = \mathbb{E}\left(\sum_{i=1}^{\tau} \underline{F}^{\tau-i} \underline{H} |\epsilon_i|\right)$, which is bounded (by Assumption 1). We have

$$\mathbb{E}(W(X_\tau) \mid X_0 = \boldsymbol{x}) \leq 1 + \|\underline{F}^\tau \boldsymbol{x}\| + \mathbb{E}\left(\sum_{i=1}^{\tau} \underline{F}^{\tau-i} \underline{H} |\varepsilon_i|\right) \leq 1 + c\|\boldsymbol{x}\| + K.$$

For $0 < \delta < 1 - c$, the set

$$D = \left\{ \boldsymbol{x} \in \mathbb{R}^{pS} \;\middle|\; \|\boldsymbol{x}\| \leq \frac{K+\delta}{1-c-\delta} \right\}$$

is compact because $K + \delta > 0$ (under Assumption 1) and $1 - c - \delta > 0$. Therefore,

$$W(\boldsymbol{x}) \geq 1, \quad \forall \boldsymbol{x} \in D, \qquad \mathbb{E}(W(X_\tau) \mid X_0 = \boldsymbol{x}) \leq (1-\delta) W(\boldsymbol{x}), \quad \forall \boldsymbol{x} \notin D.$$

Then X_τ is geometrically ergodic. □

Remark 2. *For the case $p = 1$, i.e., the restricted PEXPAR(1) model, the condition is*

$$\prod_{i=1}^{S} \Big(\max(|\phi_{i,1}|, |\phi_{i,1} + \pi_{i,1}|) \Big) < 1.$$

This is the standard result for $S = 1$. On the other hand, if we assume that the nonlinear parameter is time-dependent, we will still find the same condition because the proof is independent of this assumption.

2.3 Statistical Study

2.3.1 Quasi-Maximum Likelihood Estimation

We introduce the conditional mean and conditional variance of the model as follows:

$$\mu_t = \mathbb{E}(Z_t \mid \mathcal{F}_{t-1}), \qquad \delta_t^2 = \mathrm{Var}(Z_t \mid \mathcal{F}_{t-1}),$$

where $\mathcal{F}_{t-1} = \sigma\{Z_{t-s}, s \geq 1\}$. Then

$$\mu_t = \sum_{j=1}^{p} \big(\phi_{t,j} + \pi_{t,j} e^{-\gamma Z_{t-1}^2}\big) Z_{t-j}, \qquad \delta_t^2 = \mathbb{E}\big((Z_t - \mu_t)^2 \mid \mathcal{F}_{t-1}\big) = \sigma_t^2.$$

While the conditional mean μ_t depends on the past, the volatility δ_t^2 is periodically constant. In other words, for a given period, it does not depend on the past, reflecting the assumption of a fixed periodic structure for the conditional variance.

To estimate the parameters in each season, we denote $t = i + S\tau$, $i = 1, 2, \ldots, S$ and $\tau \in \mathbb{Z}$. This notation allows us to rewrite time t in terms of the year τ and season i. Equation (1) can then be rewritten:

$$Z_{i+S\tau} = \sum_{j=1}^{p} \big(\phi_{i,j} + \pi_{i,j} e^{-\gamma Z_{i+S\tau-1}^2}\big) Z_{i+S\tau-j} + \varepsilon_{i+S\tau}, \qquad i = 1, \ldots, S, \ \tau \in \mathbb{Z}.(eq:model) \quad (2.5)$$

Let

$$\underline{\phi}_i = (\phi_{i,1}, \pi_{i,1}, \ldots, \phi_{i,p}, \pi_{i,p})', \quad i = 1, \ldots, S, \qquad \underline{\phi} = (\underline{\phi}_1', \ldots, \underline{\phi}_S')' \in \mathbb{R}^{2pS}.$$

Assumption 2 (A1)**.** *We suppose that the process is strictly periodically stationary, i.e., condition (2.4) is satisfied.*

Assumption 3 (A2)**.** *The periodically white noise $\{\varepsilon_t, t \in \mathbb{Z}\}$ is such that $\mathbb{E}(\varepsilon_t^4) < \infty$, for any $t \in \mathbb{Z}$, which implies that $\mathbb{E}(Z_t^4) < \infty$.*

The Gaussian QML log-likelihood is given by

$$L_n(\underline{\phi}, Z_1, \ldots, Z_n) = -\frac{mS}{2}\log(2\pi) - \frac{m}{2}\sum_{i=1}^{S}\log(\sigma_i^2) - \sum_{i=1}^{S}\sum_{\tau=0}^{m-1} \frac{\big(Z_{S\tau+i} - \sum_{j=1}^{p}(\phi_{i,j} + \pi_{i,j} e^{-\gamma Z_{S\tau+i-1}^2}) Z_{S\tau+i-j}\big)^2}{2\sigma_i^2},$$

assuming $\sigma_i \neq 0$ and $n = Sm$, that is, the sample size is a multiple of the period, with Z_0 given.

The QML estimator is defined by

$$\hat{\underline{\phi}} = \arg\max L_n(\underline{\phi}, Z_1, \ldots, Z_n) = \arg\min Q_n(\underline{\phi}),$$

where

$$Q_n(\underline{\phi}) = \frac{1}{S}\sum_{i=1}^{S} Q_{i,m}(\phi_i), \qquad Q_{i,m}(\underline{\phi}_i) = \frac{1}{m}\sum_{\tau=0}^{m-1}\Big(Z_{S\tau+i} - \sum_{j=1}^{p}(\phi_{i,j} + \pi_{i,j}e^{-\gamma Z^2_{S\tau+i-1}})Z_{S\tau+i-j}\Big)^2.$$

Thus, the conditional QML estimator is exactly the least squares estimator (LSE). The solution for a fixed period i is given by

$$\hat{\underline{\phi}}_i = M_i^{-1}b_i, \tag{2.6}$$

and

$$\hat{\sigma}_i^2 = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(Z_{S\tau+i} - \sum_{j=1}^{p}\left(\varphi_{i,j} + \pi_{i,j}e^{-\gamma Z^2_{S\tau+i-1}}\right)Z_{S\tau+i-j}\right)^2. \tag{2.7}$$

Here, the vector b_i is defined as

$$b_i = \begin{pmatrix} \sum_{\tau=1}^{m-1} Z_{S\tau+i-1}Z_{S\tau+i} \\ \sum_{\tau=1}^{m-1} Z_{S\tau+i-1}Z_{S\tau+i}e^{-\gamma Z^2_{S\tau+i-1}} \\ \vdots \\ \sum_{\tau=1}^{m-1} Z_{S\tau+i-p}Z_{S\tau+i} \\ \sum_{\tau=1}^{m-1} Z_{S\tau+i-p}Z_{S\tau+i}e^{-\gamma Z^2_{S\tau+i-1}} \end{pmatrix},$$

and the matrix M_i has blocks $M_{i,j,k}$, for $j,k = 1,\dots,p$, of the form

$$M_{i,j,k} = \begin{pmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-j}Z_{S\tau+i-k} & \sum_{\tau=0}^{m-1} Z_{S\tau+i-j}Z_{S\tau+i-k}e^{-\gamma Z^2_{S\tau+i-1}} \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-j}Z_{S\tau+i-k}e^{-\gamma Z^2_{S\tau+i-1}} & \sum_{\tau=0}^{m-1} Z_{S\tau+i-j}Z_{S\tau+i-k}e^{-2\gamma Z^2_{S\tau+i-1}} \end{pmatrix}.$$

Theorem 3. *Under Assumptions A1 and A2, the QML estimators (2.6) and (2.7) are strongly consistent, and for $i = 1,\dots,S$,*

$$\sqrt{m}(\hat{\underline{\phi}}_i - \underline{\phi}_i) \xrightarrow{d} \mathcal{N}(\underline{0}_{2p},\, \sigma_i^2\Gamma_i^{-1}), \tag{2.8}$$

where

$$\Gamma_i = \begin{pmatrix} \Gamma_{i,1,1} & \cdots & \Gamma_{i,1,p} \\ \vdots & \ddots & \vdots \\ \Gamma_{i,p,1} & \cdots & \Gamma_{i,p,p} \end{pmatrix},$$

with

$$\Gamma_{i,j,k} = \begin{pmatrix} \mathbb{E}(Z_{i-j}Z_{i-k}) & \mathbb{E}(Z_{i-j}Z_{i-k}e^{-\gamma Z^2_{i-1}}) \\ \mathbb{E}(Z_{i-j}Z_{i-k}e^{-\gamma Z^2_{i-1}}) & \mathbb{E}(Z_{i-j}Z_{i-k}e^{-2\gamma Z^2_{i-1}}) \end{pmatrix}, \quad j,k = 1,\dots,p.$$

Proof. 1. **Linearity in parameters.** For each season i, the restricted PEXPAR(p) regression is *linear in the unknown parameters* (the exponential term is data-driven and multiplies lagged variables, not parameters). Hence the criterion is quadratic and the normal equations are linear.

2. **QML = LS in this setting.** Under the Gaussian quasi-likelihood (with season-specific but fixed variances), maximizing the conditional log-likelihood is *equivalent* to minimizing the sum of squared residuals. Therefore, the QML estimator coincides exactly with the least-squares (LS) estimator season by season.

3. **Asymptotic properties carry over.** Under Assumptions A1–A2 (strict periodic stationarity and finite fourth moments) and standard identification/regularity conditions, the established LS results for the restricted PEXPAR model (consistency and asymptotic normality) directly *transfer* to QML because the two estimators are identical here; see [17] for LS and, for general QML-to-LS equivalence and asymptotics in nonlinear time-series regressions, [10, 2].

□

Remark 3. *For the selection of the order p, we can use the Akaike information criterion (AIC) defined by*

$$AIC(p) = m \sum_{i=1}^{S} \log \hat{\sigma}_i^2 + 2(2pS).$$

This criterion balances goodness-of-fit, through the log-likelihood term, and model complexity, through the penalty term, in order to avoid overfitting.

2.3.2 Parameter tests

Tests for the Nullity of the Last Coefficient: PEXPAR(p) vs. PEXPAR(p-1)

In this section, we address the problem of testing the significance of the parameters, with a particular focus on the nullity test of the last coefficient. We will employ the Likelihood Ratio (LR) test and leverage the asymptotic normality distribution obtained in the theorem.

It is worth noting that the LR test examines whether certain coefficients are equal to zero, representing an interior point within the parameter space, rather than being on the boundary. Additionally, we will examine the same problem using the Lagrange Multiplier (LM) test. The detailed proofs are provided in Appendix A.

Likelihood Ratio Test. We are interested in testing hypotheses of the form:

$$H_0 : (\phi_{i,p}, \pi_{i,p})' = 0, \quad \forall i = 1, \ldots, S \qquad \text{vs.} \qquad H_1 : \exists i \text{ such that } (\phi_{i,p}, \pi_{i,p})' \neq 0.$$

Under H_1, the model is the restricted PEXPAR(p). The QML estimator $\hat{\underline{\phi}}_i$ is given by (2.6) with $Q_{i,m}(\hat{\underline{\phi}}_i)$ given by (2.7).

Under H_0, the model is the restricted PEXPAR($p-1$). The QML estimator is

$$\tilde{\underline{\phi}}_i = (\phi_{i,1}, \pi_{i,1}, \ldots, \phi_{i,p-1}, \pi_{i,p-1}, 0, 0)' = (\tilde{\underline{\phi}}_i^*, 0, 0)',$$

where $\tilde{\underline{\phi}}_i^*$ is given by (6) with the order p replaced by $p-1$, and

$$Q_{i,m}(\tilde{\underline{\phi}}_i) = \frac{1}{m} \sum_{\tau=0}^{m-1} \left(Z_{S\tau+i} - \sum_{j=1}^{p-1} \left(\tilde{\phi}_{i,j} + \pi_{i,j} e^{-\gamma Z^2_{S\tau+i-1}} \right) Z_{S\tau+i-j} \right)^2.$$

Theorem 4. *Under the hypotheses H_0 and H_1, the likelihood ratio test statistic*

$$LR_m = m \sum_{i=1}^{S} \log \left(\frac{Q_{i,m}(\tilde{\underline{\phi}}_i)}{Q_{i,m}(\hat{\underline{\phi}}_i)} \right)$$

asymptotically follows a χ^2 distribution with $2S$ degrees of freedom. Specifically, the test rejects H_0 at the asymptotic level α when

$$LR_m > \chi^2_{2S}(1-\alpha).$$

Lagrange Multiplier Test. For the same problem, the null hypothesis is

$$H_0 : R\underline{\phi}_i = 0 \quad \text{for all } i \qquad \text{vs} \qquad H_1 : \exists i \ / \ R\underline{\phi}_i \neq 0,$$

where

$$R = \begin{pmatrix} 0_{2\times 2(p-1)} & I_{2\times 2} \end{pmatrix}_{2\times 2p}.$$

We define the diagonal block matrix

$$\Gamma = \begin{pmatrix} \frac{1}{\sigma_1^2}\Gamma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_S^2}\Gamma_S \end{pmatrix}_{2pS\times 2pS}.$$

Then,

$$\sqrt{m}\,(\underline{\hat{\phi}} - \underline{\phi}) \xrightarrow{d} \mathcal{N}\big(\underline{0}_{2pS}, \Gamma^{-1}\big).$$

We denote the score vector as

$$G(\underline{\phi}) = \big(G_1(\underline{\phi}_1)', \ldots, G_S(\underline{\phi}_S)'\big)'_{2pS\times 1},$$

where

$$G_i(\underline{\phi}_i) = \frac{1}{\sigma_i^2}\begin{pmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}\,\varepsilon_{i+S\tau} \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}\exp(-\gamma Z_{S\tau+i-1}^2)\,\varepsilon_{i+S\tau} \\ \vdots \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}\,\varepsilon_{i+S\tau} \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}\exp(-\gamma Z_{S\tau+i-1}^2)\,\varepsilon_{i+S\tau} \end{pmatrix} = \frac{1}{\sigma_i^2}\begin{pmatrix}\boldsymbol{Z}_i & \tilde{Z}_i\end{pmatrix}'\boldsymbol{\varepsilon}_i.$$

Here,

$$\boldsymbol{Z}_i = \begin{pmatrix} Z_{i-1} & Z_{i-1}\exp(-\gamma Z_{i-1}^2) & \cdots & Z_{i-(p-1)}\exp(-\gamma Z_{i-1}^2) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{S(m-1)+i-1} & Z_{S(m-1)+i-1}\exp(-\gamma Z_{i-1}^2) & \cdots & Z_{S(m-1)+i-(p-1)}\exp(-\gamma Z_{i-1}^2) \end{pmatrix}_{m\times 2(p-1)},$$

$$\tilde{Z}_i = \begin{pmatrix} Z_{i-p} & Z_{i-p}\exp(-\gamma Z_{i-1}^2) \\ \vdots & \vdots \\ Z_{S(m-1)+i-p} & Z_{S(m-1)+i-p}\exp(-\gamma Z_{i-1}^2) \end{pmatrix}_{m\times 2},$$

and

$$\boldsymbol{\varepsilon}_i = \big(\varepsilon_i, \ldots, \varepsilon_{i+S(m-1)}\big)'_{m\times 1}.$$

Theorem 5. *Under the hypotheses H_0 and H_1, the Lagrange Multiplier test statistic follows a χ^2 distribution with $2S$ degrees of freedom. Specifically, the LM test statistic is given by*

$$LM = \sum_{i=1}^{S} \tilde{\sigma}_i^2\, G_i(\tilde{\phi}_i)'\Gamma_i^{-1}G_i(\tilde{\phi}_i) \;\sim\; \chi^2_{2S}.$$

Tests of Linearity

We now turn to the problem of testing linearity in the PEXPAR model. Before presenting the main results, it is worth noting that the proofs for the linearity tests are similar to those established for the nullity of the order-p coefficients. These proofs are provided in Appendix B.

Likelihood Ratio Test. If $\pi_{i,j} = 0,\ \forall i,\ \forall j$, then the model becomes the periodic AR(p). We make the following hypotheses:

$$H_0 : \pi_{i,j} = 0,\ \forall i,\ \forall j \qquad \text{vs.} \qquad H_1 : \exists\, i \text{ or } j\ /\ \pi_{i,j} \neq 0,$$

which is equivalent to stating

$$H_0 : Z_t \sim PAR(p) \qquad \text{vs.} \qquad H_1 : Z_t \sim PEXPAR(p).$$

Under the null hypothesis, the estimator is

$$\underline{\tilde{\phi}}_i = \left(\begin{bmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}^2 & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} \\ \vdots & \ddots & \vdots \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{\tau=1}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i} \\ \vdots \\ \sum_{\tau=1}^{m-1} Z_{S\tau+i-p} Z_{S\tau+i} \end{bmatrix} \right).$$

The residual-based quadratic form is then

$$Q_{i,m}(\underline{\tilde{\phi}}_i) = \frac{1}{m} \sum_{\tau=0}^{m-1} \left(Z_{S\tau+i} - \sum_{j=1}^{p} \tilde{\phi}_{i,j} Z_{S\tau+i-j} \right)^2.$$

Theorem 6. *Under the hypotheses H_0 and H_1, the likelihood ratio test statistic LR_m asymptotically follows a χ^2 distribution with Sp degrees of freedom. Specifically, the test rejects H_0 at the asymptotic level α when*

$$LR_m = m \sum_{i=1}^{S} \log\left(\frac{Q_{i,m}(\tilde{\phi}_i)}{Q_{i,m}(\hat{\phi}_i)} \right) \ > \ \chi^2_{Sp}(1-\alpha),$$

where $\tilde{\phi}_i$ and $\hat{\phi}_i$ denote the parameter estimates under the null and alternative hypotheses, respectively.

Lagrange Multiplier Test

Since the model is linear under H_0, it is advantageous to use the LM test because it allows us to compute the test statistic solely under the null hypothesis. To proceed, we reorder the parameters to separate the linear component from the nonlinear component. The parameter vector is reformulated using the matrix K where

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{2p\times 2p} .$$

Therefore

$$\vartheta_i = K\underline{\phi}_i = (\phi_{i,1}, \dots, \phi_{i,p}; \pi_{i,1}, \dots, \pi_{i,p})', \quad i = 1, \dots, S, \quad \text{and } \underline{\theta} = (\vartheta_1, \dots, \vartheta_S)' \in \mathbb{R}^{2pS}.$$

Remark that

$$K^{-1} = K'. \tag{2.9}$$

The null hypothesis is then

$$H_0 : R\vartheta_i = 0, \ \forall i \quad \text{vs} \quad H_1 : \exists i \,/\, R\vartheta_i \neq 0,$$

where

$$R = \begin{pmatrix} 0_{p\times p} & I_{p\times p} \end{pmatrix} .$$

From (2.8) and (2.9), we have the distribution of the estimator under H_1:

$$\sqrt{m}(\hat{\vartheta}_i - \vartheta_i) \ \xrightarrow{d} \ \mathcal{N}\big(\underline{0}_{2p}, \, \sigma_i^2 (K\Gamma_i K')^{-1}\big). \tag{2.10}$$

We set the diagonal block matrix

$$\boldsymbol{W} = \begin{pmatrix} \frac{1}{\sigma_1^2} K\Gamma_1 K' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_S^2} K\Gamma_S K' \end{pmatrix}_{2pS\times 2pS},$$

so that

$$\sqrt{m}(\underline{\hat{\theta}} - \underline{\theta}) \quad \xrightarrow{d} \quad \mathcal{N}(\underline{0}_{2pS}, \boldsymbol{W}^{-1}).$$

We note the score vector as

$$G(\underline{\theta}) = \big(G_1(\vartheta_1)', \ldots, G_S(\vartheta_S)'\big)'_{2pS\times 1},$$

where

$$G_i(\vartheta_i) = \frac{1}{\sigma_i^2}\Big(\sum_{\tau=0}^{m-1} Z_{S\tau+i-1}\varepsilon_{i+S\tau}, \ldots, \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}\varepsilon_{i+S\tau}; \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}e^{-\gamma Z^2_{S\tau+i-1}}\varepsilon_{i+S\tau}, \ldots, \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}e^{-\gamma Z^2_{S\tau+i-}}$$

Equivalently,

$$G_i(\vartheta_i) = \frac{1}{\sigma_i^2}(\boldsymbol{Z}_i,\, Z_i^*)'\varepsilon_i,$$

with

$$\boldsymbol{Z}_i = \begin{pmatrix} Z_{i-1} & \cdots & Z_{i-p} \\ \vdots & \ddots & \vdots \\ Z_{S(m-1)+i-1} & \cdots & Z_{S(m-1)+i-p} \end{pmatrix}_{m\times p}, \quad Z_i^* = \begin{pmatrix} Z_{i-1}e^{-\gamma Z^2_{i-1}} & \cdots & Z_{i-p}e^{-\gamma Z^2_{i-1}} \\ \vdots & \ddots & \vdots \\ Z_{S(m-1)+i-1}e^{-\gamma Z^2_{i-1}} & \cdots & Z_{S(m-1)+i-p}e^{-\gamma Z^2_{i-1}} \end{pmatrix}_{m\times p}$$

and $\varepsilon_i = (\varepsilon_i, \ldots, \varepsilon_{i+S(m-1)})'_{m\times 1}$.

Theorem 7 (LM Test for Linearity)**.** *Under the hypotheses H_0 and H_1, the Lagrange Multiplier test statistic follows a χ^2 distribution with pS degrees of freedom. Specifically,*

$$LM = \sum_{i=1}^{S} \tilde{\sigma}_i^2 G_i(\tilde{\vartheta}_i)'(K\tilde{\Gamma}_i K')^{-1} G_i(\tilde{\vartheta}_i) \quad \overset{d}{\sim} \quad \chi^2_{pS}.$$

Proofs for these results are provided in Appendix B.

2.4 Simulation Results and Application

2.4.1 Simulation Results

To better understand the properties of PEXPAR models, we generated three different time series, each consisting of 80,000 observations with a period of 2 and order 1. Our simulations reveal phenomena consistent with those described by [22] p. 191) for the EXPAR model. Specifically, when variances are fixed at $(0.02, 0.08)$, the generated time series exhibit distinct non-Gaussian marginal distributions. These are characterized by features such as sharp peaks, heavy tails, or bimodality, depending on the model parameters (see Figure 2.1). These results underscore the flexibility of the periodic EXPAR model in capturing a wide range of marginal density shapes, determined by the choice of its parameters.

Model A.

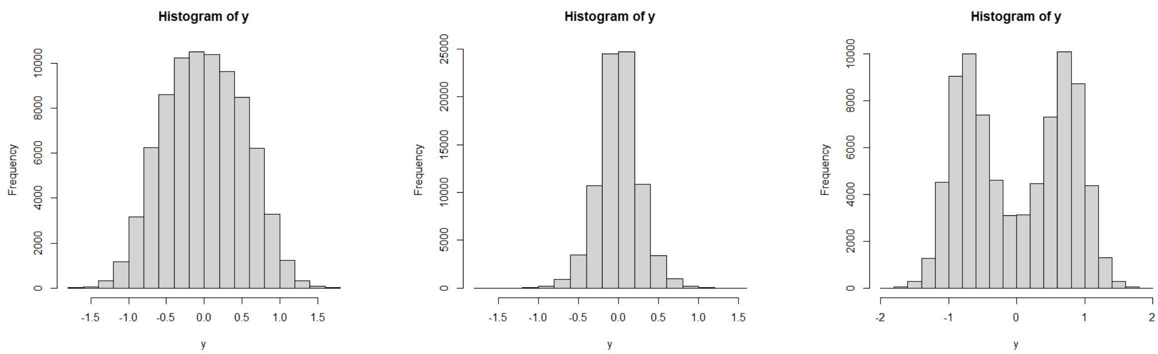
$$\begin{cases} Z_{2\tau+1} = \left(0.8 + 0.2e^{-Z_{2\tau}^2}\right) Z_{2\tau} + \varepsilon_{2\tau+1}, \\ \\ Z_{2\tau+2} = \left(0.6 + 0.4e^{-0.5Z_{2\tau+1}^2}\right) Z_{2\tau+1} + \varepsilon_{2\tau+2}. \end{cases}$$

Model B.

$$\begin{cases} Z_{2\tau+1} = \left(1 - 0.8e^{-Z_{2\tau}^2}\right) Z_{2\tau} + \varepsilon_{2\tau+1}, \\ \\ Z_{2\tau+2} = \left(1 - 0.4e^{-0.5Z_{2\tau+1}^2}\right) Z_{2\tau+1} + \varepsilon_{2\tau+2}. \end{cases}$$

Model C.

$$\begin{cases} Z_{2\tau+1} = \left(0.7 + 0.5e^{-Z_{2\tau}^2}\right) Z_{2\tau} + \varepsilon_{2\tau+1}, \\ \\ Z_{2\tau+2} = \left(0.6 + 0.8e^{-Z_{2\tau+1}^2}\right) Z_{2\tau+1} + \varepsilon_{2\tau+2}. \end{cases}$$



Model A **Model B** **Model C**

Figure 2.1: Histograms for PEXPAR models.

2.4.2 Testing the Nullity of the Last Coefficients

Now, we proceed to test the nullity of the last coefficients. For this purpose, we simulate the following models:

Model I. Restricted $\text{PEXPAR}_2(1)$ with parameters

$$\phi = (-0.8,\, 0.6;\, 0.5,\, -0.4)'.$$

Model II. Restricted $\text{PEXPAR}_2(2)$ with parameters

$$\phi = (-0.8,\, 0.6;\, 0.5,\, -0.4;\, 0.4,\, -0.7;\, -0.5,\, 0.8)'.$$

Model III. Periodic Autoregressive model, $\text{PAR}_2(2)$, with parameters

$$\phi = (-0.8,\, 0.5;\, 0.4,\, -0.5)'.$$

Model IV. Periodic Autoregressive model, $\text{PAR}_4(1)$, with parameters

$$\phi = \begin{pmatrix} -0.8 & 0.5 \\ 0.6 & -0.4 \\ -0.5 & 0.3 \\ 0.4 & -0.6 \end{pmatrix}.$$

Model V. Periodic Exponential Autoregressive model, $\text{PEXPAR}_4(2)$, with parameters

$$\phi = \begin{pmatrix} -0.8 & 0.3 & 0.5 & -0.7 \\ 0.6 & -0.5 & -0.4 & 0.8 \\ -0.5 & 0.7 & 0.3 & -0.6 \\ 0.4 & -0.8 & -0.6 & 0.5 \end{pmatrix}.$$

The parameter values were randomly selected while ensuring the stationarity of the models under both H_0 and H_1. The error terms follow a normal distribution, and the variances are set to $(1,1)'$ for the first three models and $(1, 0.9, 1.1, 1)'$ for the others. In all cases, the parameter $\gamma = 1$ was selected, as it represents a reasonable middle value, neither too small (< 1) nor too large (> 1). This choice simplifies the comparison between the null and alternative hypotheses and allows the focus to remain on testing the influence of other parameters while keeping γ constant. For the likelihood ratio (LR) test, we estimate the models under both H_0 and H_1. In contrast, for the Lagrange Multiplier (LM) test, only the model under H_0 is estimated, consistent with the theoretical framework. Model I, reflecting the model underH_0,is used to calculate the level, while Model II, representing the model underH_1, is chosen to calculate the power. Table 2.1 presents the rejection rates for the LR and LM tests at the 5% and 10% significance levels. The asymptotic distribution of LR_m under the null hypothesis with $n = 200$ is depicted in Figure 2.2. According to Table2.1, the power increases with the sample size n, and both tests maintain reasonably well-controlled levels. When the model is stable, the results hold whether we consider all positive or negative parameters (see Table 2.2 and Table2.3, where bold numbers cor respond to the alternative hypothesis). Figure 2.2 illustrates that the LR distribution closely approximates the theoretical χ_4^2 distribution. Additionally, we observe that the LR test tends to over-reject the null hypothesis for small sample sizes but becomes asymptotically equivalent to the LM test for large sample sizes.

Table 2.1: Rejection frequencies computed on 1000 replications.

Model	α	$n = 100$	$n = 200$	$n = 300$	$n = 500$
I (LR)	5%	0.072	0.061	0.058	0.056
	10%	0.143	0.106	0.100	0.107
II (LR)	5%	0.514	0.808	0.949	0.997
	10%	0.604	0.867	0.971	0.999
I (LM)	5%	0.057	0.056	0.053	0.053
	10%	0.128	0.096	0.094	0.103
II (LM)	5%	0.459	0.797	0.946	0.997
	10%	0.574	0.861	0.968	0.999

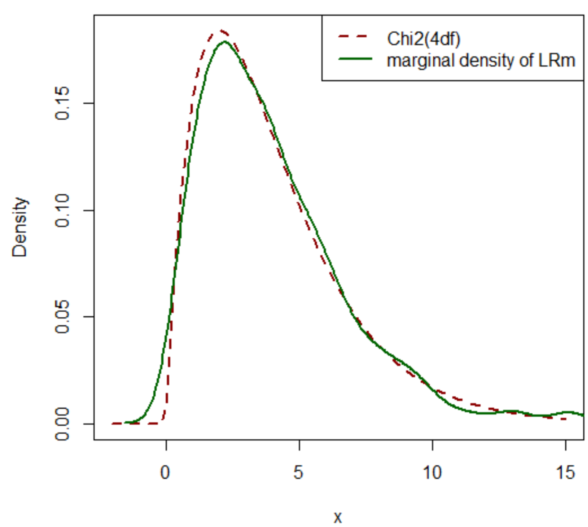


Figure 2.2: Asymptotic distribution of LR_m under H_0 with $n = 200$.

Table 2.2: Rejection frequencies computed on 1000 replications with $\phi = (0.6, 0.2;\ \mathbf{0.2, 0.1};\ 0.4, 0.3;\ \mathbf{0.1, 0.2})'$.

Model	α	$n = 100$	$n = 200$	$n = 300$	$n = 500$
I (LR)	5%	0.071	0.066	0.057	0.050
	10%	0.137	0.122	0.110	0.106
II (LR)	5%	0.771	0.978	0.998	1.000
	10%	0.854	0.989	1.000	1.000
I (LM)	5%	0.058	0.056	0.055	0.046
	10%	0.112	0.114	0.099	0.100
II (LM)	5%	0.739	0.974	0.997	1.000
	10%	0.834	0.988	1.000	1.000

Table 2.3: Rejection frequencies computed on 1000 replications with $\phi = (-0.9, -1;\ \mathbf{-0.4, -0.2};\ -0.3, -0.1;\ \mathbf{-0.1, -0.2})'$.

Model	α	$n = 100$	$n = 200$	$n = 300$	$n = 500$
I (LR)	5%	0.081	0.062	0.059	0.053
	10%	0.133	0.111	0.105	0.102
II (LR)	5%	0.825	0.979	0.999	1.000
	10%	0.901	0.994	1.000	1.000
I (LM)	5%	0.061	0.055	0.054	0.053
	10%	0.116	0.105	0.099	0.098
II (LM)	5%	0.790	0.976	0.999	1.000
	10%	0.887	0.994	1.000	1.000

Test for Linearity

We employ Model III to test for linearity. The same observations discussed above apply here as well, as shown in Table2.4 and Figure 2.3.

Table 2.4: The rejection frequency computed on 1000 replications.

Model	α	$n = 100$	$n = 200$	$n = 300$	$n = 500$
III (LR)	5%	0.084	0.066	0.058	0.047
	10%	0.147	0.124	0.116	0.099
II (LR)	5%	0.717	0.968	0.998	1
	10%	0.826	0.984	0.999	1
III (LM)	5%	0.069	0.057	0.054	0.046
	10%	0.121	0.111	0.110	0.094
II (LM)	5%	0.685	0.974	0.998	1
	10%	0.802	0.982	0.999	1

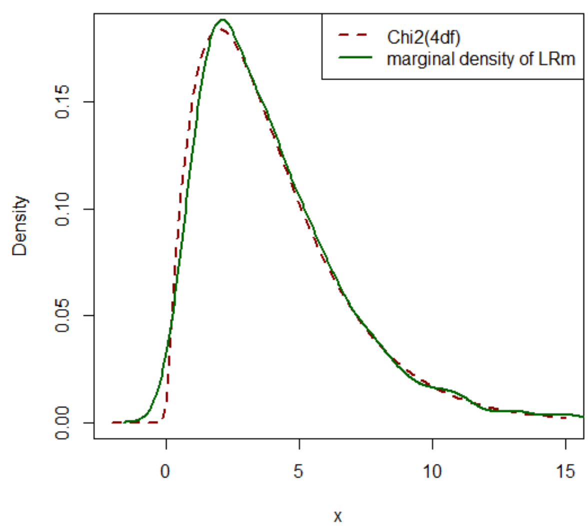


Figure 2.3: Asymptotic distribution of LR_m, $n = 500$.

To further explore this, we increase model complexity and perform a linearity test comparing Model IV ($\text{PAR}_4(2)$) against Model V ($\text{PEXPAR}_4(2)$). As shown in Table 2.5, the levels are accurately estimated, and the power increases with the sample size n. Figure 2.4

illustrates that the empirical and theoretical distributions of the LR test overlap closely with χ^2_8.

Table 2.5: The rejection frequency computed on 1000 replications.

Model	α	$n = 400$	$n = 800$
LR(H_0)	5%	0.074	0.055
	10%	0.136	0.114
LR(H_1)	5%	0.992	1
	10%	0.997	1
LM(H_0)	5%	0.060	0.054
	10%	0.124	0.112
LM(H_1)	5%	0.992	1
	10%	0.996	1

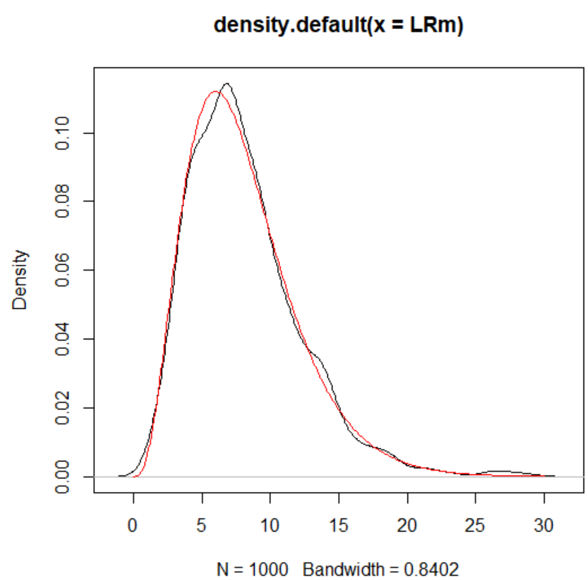


Figure 2.4: Asymptotic distribution of LR_m, $n = 800$.

2.4.3 Application: Modeling the Rainfall in Algeria

Understanding and modeling rainfall patterns is of paramount importance in various fields such as agriculture, water resource management, and climate science. Algeria, with its diverse landscape and significant agricultural sector, relies heavily on seasonal rainfall for its economic activities and the livelihoods of its people. Moreover, as climate change continues to impact global weather patterns, accurate modeling of rainfall becomes increasingly vital for predicting and mitigating potential risks associated with extreme weather events.

In this application, we focus on modeling the monthly rainfall series of Algeria spanning from January 1901 to December 2015. By analyzing this dataset, we aim to gain insights into the temporal variations, trends, and statistical characteristics of rainfall in Algeria. This endeavor is crucial not only for understanding historical rainfall patterns but also for developing robust models that can provide valuable forecasts for future rainfall scenarios. Through our analysis, we seek to contribute to the broader understanding of regional climate dynamics and support informed decision-making in various sectors reliant on rainfall data. The dataset is obtained from the World Bank Climate Knowledge Portal[1]. Figure 2.5 shows part of the data. In Table 2.6, descriptive statistics provide insights into the time series:

- The average rainfall in Algeria is approximately 6.94 mm. It is important to note that 1 millimeter of rainfall represents 1 liter of water per square meter.

- The variability of rainfall around the mean is relatively high, with a standard deviation of approximately 4.53 mm.

- The lowest recorded rainfall is 0.1698 mm, observed in December 2015, indicating periods of very low precipitation.

- The highest recorded rainfall is 25.1144 mm, observed in January 1904, suggesting instances of heavy rainfall.

[1] `https://climateknowledgeportal.worldbank.org/download-data`

- The difference between the maximum and minimum values is 24.9445 mm, highlighting the wide variability in rainfall intensity.

- The skewness value of 0.8604 suggests that the distribution of rainfall is moderately positively skewed.

- The excess kurtosis value of 0.4015 indicates that the distribution of rainfall is more peaked and has heavier tails compared to a normal distribution.

It is important to note that the minimum was recorded more recently than the maximum, which may reflect the impact of climate change. The combination of positive skewness and higher kurtosis suggests that while extreme values are more likely to occur, they are primarily concentrated on the right tail of the distribution, leading to a higher peak around the mean. From the skewness and kurtosis values, it is evident that the rainfall data notably diverges from a normal distribution. This divergence is further supported by the Jarque-Bera test, where the p-value is less than 2.2×10^{-16}.

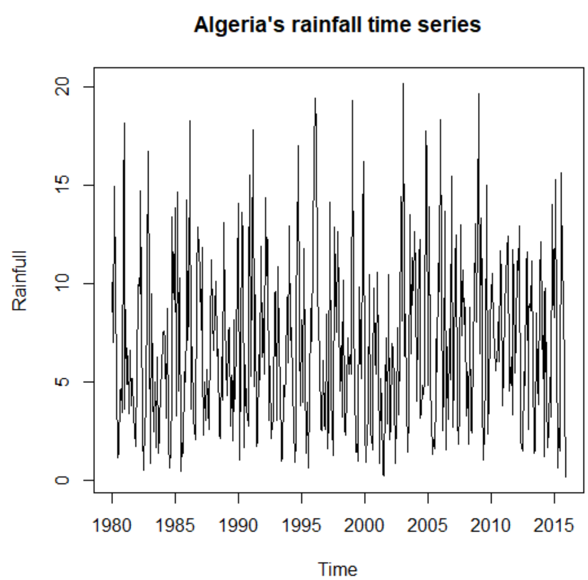


Figure 2.5: Monthly rainfall series in Algeria (1901–2015).

Table 2.6: Descriptive statistics of the monthly rainfall in Algeria (1901–2015).

Statistic	Value
Mean	6.9400
Standard deviation	4.5300
Minimum (Dec 2015)	0.1698
Maximum (Jan 1904)	25.1144
Range	24.9445
Skewness	0.8604
Excess kurtosis	0.4015
Jarque-Bera p-value	$< 2.2 \times 10^{-16}$

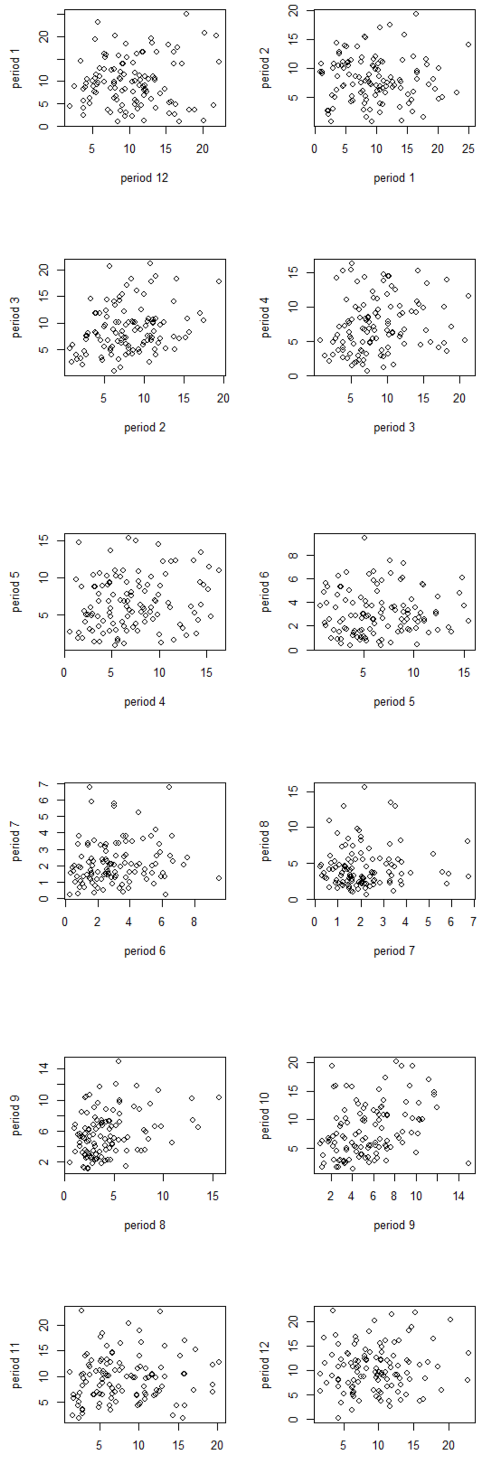


Figure 2.6: Scatter plots of the rainfall time series.

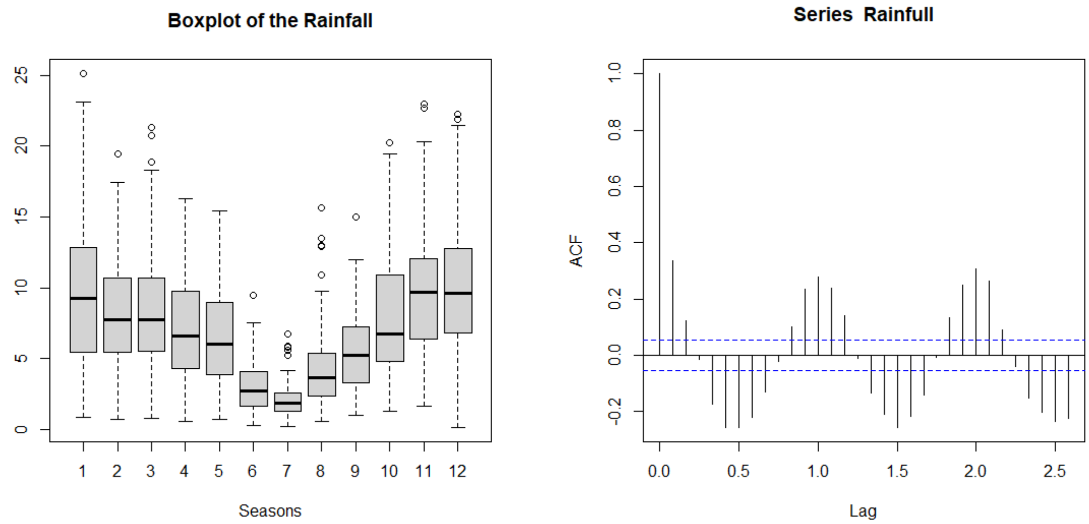


Figure 2.7: Boxplot (left) and ACF (right) of the rainfall time series.

The periodic nature of this time series is evident in the scatter plots depicted in Figure 2.6. Strong correlations are observed between periods 9 and 10, periods 8 and 9, periods 3 and 4, and periods 2 and 3. Figure 2.7, displaying the boxplot, illustrates fluctuations in the series' dynamics across seasons, as indicated by varying averages. Further analysis reveals periodicity in the autocorrelation function (ACF), with positive correlations occurring at lag multiples of 12. To facilitate analysis, the data were log-transformed and centered.

Finally, the Keenan test conducted on the log-transformed rainfall data yielded a p-value of 4.935764×10^{-11}, providing compelling evidence for the nonlinearity of this time series.

Model Selection The goal is to model the given time series using the restricted PEXPAR model. Initial analyses begin with nonlinear parameter estimation for $\epsilon = 10^{-13}$, resulting in $\gamma = 5.64$. To determine the appropriate order, we performed the LR test comparing restricted PEXPAR$_{12}$(1) with restricted PEXPAR$_{12}$(2). The test statistic, $LR_m = 29.4424$, falls below the critical value $\chi^2_{24}(0.95) = 36.4150$, leading to the acceptance of H_0 and favoring PEXPAR$_{12}$(1).

The model's validity was further assessed using the likelihood ratio test for linearity, yielding $LR_m = 28.2784$. This exceeds the critical value $\chi^2_{12}(0.95) = 21.0260$, rejecting linearity in favor of the restricted $\text{PEXPAR}_{12}(1)$ model.

Table 2.7 illustrates the AIC values for competing models, where $\text{PAR}_{12}(1)$ outperforms in simplicity, but restricted $\text{PEXPAR}_{12}(1)$ provides superior modeling accuracy for nonlinearity. The $\text{PAR}_{12}(1)$ model's periodic orders (Table 2.8) were determined based on the Bayesian Information Criterion (BIC), resulting in

$$\text{PAR}_{12}(0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0).$$

Table 2.9 reports the parameter estimates for $\text{PAR}_{12}(1)$, with non-zero coefficients statistically significant.

Table 2.7: AIC criterion.

Model	$\text{PAR}_{12}(1)$	$\text{PEXPAR}_{12}(1)$
AIC	-896.4741	-840.2187

Table 2.8: Orders of the $\text{PAR}_{12}(1)$.

Periods	1	2	3	4	5	6	7	8	9	10	11	12
Orders	0	0	1	1	0	0	1	0	1	1	0	0

Diagnostic Evaluation Following McLeod's (1994) approach, diagnostic checks on PAR residuals revealed significant non-normality (Table 2.11) despite non-correlation at the 5% significance level (Table 2.10). Consequently, the $\text{PEXPAR}_{12}(1)$ model was explored further. Table 2.12 summarizes the parameter estimates and standard errors for restricted $\text{PEXPAR}_{12}(1)$. Residual diagnostics (Table 2.13) indicate a generally good fit. Residual correlations at period 8 suggest unmodeled dynamics, though residuals are uncorrelated at a stricter 1% level for all periods.

Residual variances (Table 2.14) reveal marginal improvements with $\text{PEXPAR}_{12}(1)$ compared to $\text{PAR}_{12}(1)$, affirming its suitability for capturing nonlinear structures.

Table 2.9: Parameter estimation and standard errors for $PAR_{12}(1)$.

Period	1	2	3	4	5	6	7	8	9	10	11	12
ϕ_i	–	–	0.2947	0.2679	–	–	0.2067	–	0.3005	0.3693	–	–
S.E	–	–	0.0798	0.1034	–	–	0.0841	–	0.0849	0.0959	–	–

Table 2.10: Box-Ljung test for $PAR_{12}(1)$ residuals.

Period	1	2	3	4	5	6	7	8	9	10	11	12
p-value	.5219	.1302	.0618	.138	.5491	.9067	.5813	.427	.8141	.2787	.3569	.5793

Table 2.11: Jarque-Bera test for $PAR_{12}(1)$ residuals.

Period	1	2	3	4	5	6	7	8	9	10	11	12
p-value	.0000	.0000	.0000	.0001	.0027	.0098	.0090	.9149	.0703	.2741	.0008	.0000

Table 2.12: Parameters of restricted $PEXPAR_{12}(1)$ model.

Period	1	2	3	4	5	6	7	8	9	10	11	12
$\phi_{i,1}$	.1273	.0448	.2938	.2141	.1575	-.0118	.1942	.0206	.2522	.3510	.1493	.0862
S.E	.0056	.0058	.0040	.0047	.0059	.0056	.0067	.0042	.0045	.0059	.0055	.0065
$\pi_{i,1}$	-.6751	-.5052	-.3070	.5727	.3757	.9906	-.0971	-.3516	.4893	-.0636	-.4371	.0285
S.E	.0290	.0369	.0219	.0219	.0326	.0343	.0432	.0236	.0224	.0283	.0323	.0260

Table 2.13: Box-Ljung test for $PEXPAR_{12}(1)$ residuals.

Period	1	2	3	4	5	6	7	8	9	10	11	12
p-value	.8076	.8666	.8842	.9128	.2513	.2589	.2093	.0411	.2545	.5686	.0831	.0847

Table 2.14: Residual variances for $\mathrm{PAR}_{12}(1)$ and restricted $\mathrm{PEXPAR}_{12}(1)$ models.

Period	1	2	3	4	5	6	7	8	9	10	11	12
PAR	.4758	.3747	.2749	.3785	.4237	.4888	.3983	.3525	.2927	.3436	.2750	.3680
PEXPAR	.4717	.3671	.2735	.3752	.4054	.4785	.3954	.3508	.2880	.3371	.2662	.3622

Forecasting Performance One-step-ahead forecasts for 2016 were conducted using both models. Figure 2.8 illustrates that the $\mathrm{PEXPAR}_{12}(1)$ predictions align closely with the observed data, effectively capturing the fluctuations and overall patterns, and outperforming the $\mathrm{PAR}_{12}(1)$ model.

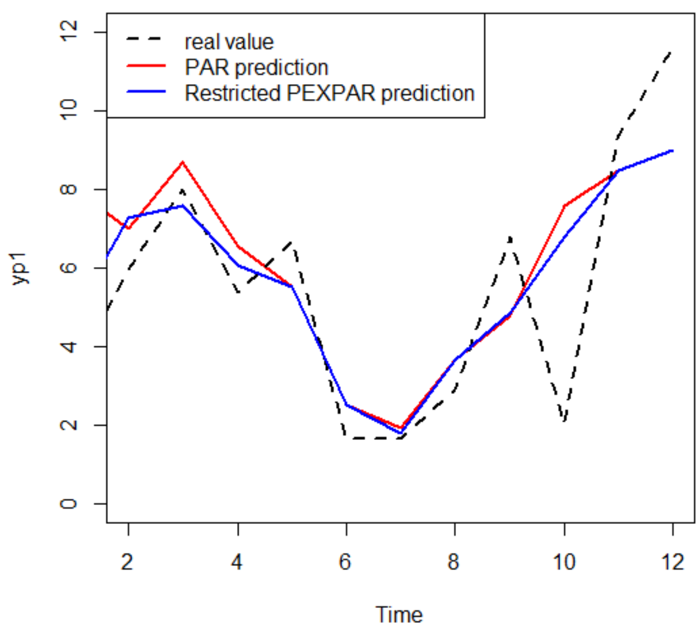


Figure 2.8: Comparison of real values with $\mathrm{PAR}_{12}(1)$ and restricted $\mathrm{PEXPAR}_{12}(1)$ predictions.

2.5 Wald Tests

In this section, we present two Wald-type tests for the Restricted PEXPAR(p) model. The first test concerns the nullity of the last autoregressive coefficient, which allows to determine the order of the model. The second test deals with the problem of testing linearity against the alternative of a nonlinear specification.

2.5.1 Test for the Nullity of the Last Coefficient

We consider the same problem of testing the significance of the last coefficient. Formally, the hypotheses are the same as before, and we keep the same notation.

The Wald statistic is given by

$$W_m = m(\boldsymbol{R}\underline{\hat{\phi}})' \left(\boldsymbol{R}\Gamma^{-1}\boldsymbol{R}'\right)^{-1} (\boldsymbol{R}\underline{\hat{\phi}}),$$

and the null hypothesis H_0 is rejected at the asymptotic level α whenever

$$W_m > \chi^2_{2S}(1-\alpha).$$

In applications, Γ must be replaced by a consistent estimator.

2.5.2 Test for Linearity in the Restricted PEXPAR(p) Model

To test for linearity, it is convenient to rearrange the parameters so that the nonlinear coefficients are placed at the end. For this purpose, we use the same matrix K and parameter vector θ as defined previously.

The Wald statistic for testing linearity is

$$W_m = m\,(\boldsymbol{R}\hat{\theta})' \left(\boldsymbol{R}\boldsymbol{V}^{-1}\boldsymbol{R}'\right)^{-1} (\boldsymbol{R}\hat{\theta}),$$

and the null hypothesis H_0 is rejected at the asymptotic level α whenever

$$W_m > \chi^2_{Sp}(1-\alpha).$$

Remark 4. *Rejecting H_0 in this test provides evidence of nonlinear effects, indicating that the linear restricted specification is not sufficient to capture the dynamics of the series.*

2.5.3 Simulation Results

To assess the finite-sample performance of the Wald tests, we conducted a Monte Carlo study based on 1000 replications for different sample sizes ($n = 100, 300, 500$).

Test for the Nullity of the Last Coefficient

We considered two models under the Restricted PEXPAR specification with $S = 2$:

Model I: $p = 1$, parameters $\phi = (0.9, 0.5; 0.3, 0.8)$.

Model II: $p = 2$, parameters $\phi = (0.9, 0.5; 0.3, 0.8; 0.2, 0.6; 0.5, 0.7)$.

The rejection frequencies of the Wald test at nominal levels 5% and 10% are reported in Table 2.15.

Table 2.15: Rejection frequencies of the Wald test for the last coefficient based on 1000 replications.

Model	α	$n = 100$	$n = 300$	$n = 500$
I	5%	0.088	0.068	0.052
	10%	0.160	0.106	0.103
II	5%	0.760	0.998	1.000
	10%	0.864	0.999	1.000

Interpretation. For Model I, which corresponds to the null hypothesis, the empirical rejection frequencies are close to the nominal levels, particularly for larger samples. This indicates that the Wald test controls the type I error adequately. For Model II, where the null is false, the rejection frequencies increase rapidly and approach one even for moderate sample sizes. This shows that the test has excellent power to detect a nonzero last coefficient.

Test for Linearity

We next examined the test for linearity in the Restricted PEXPAR(2) model with $S = 2$. Two parameter settings were considered:

- $\{(0.9, -0.5), (-0.4, 0.7)\}$,
- $\{(0.9, -0.5), (-1, 0.8), (-0.4, 0.7), (0.6, -1.2)\}$.

The rejection frequencies of the Wald test for linearity are reported in Table 2.16.

Table 2.16: Rejection frequencies of the Wald test for linearity based on 1000 replications.

Model	α	$n = 100$	$n = 300$	$n = 500$
I	5%	0.082	0.065	0.054
	10%	0.147	0.117	0.107
II	5%	0.854	0.999	1.000
	10%	0.890	1.000	1.000

Interpretation. For Model I (linear specification), the rejection rates are close to the nominal levels when the sample size increases, but there is a tendency to slightly over-reject for small samples ($n = 100$). This suggests that the linearity test is somewhat oversized in small samples. For Model II (nonlinear specification), the test exhibits very high rejection frequencies, approaching one quickly as n increases. This confirms that the test is powerful for detecting nonlinear dynamics in the Restricted PEXPAR model.

Chapter 3

Recursive Estimation in Restricted Exponential Autoregressive Models

3.1 Introduction

The most standard estimation technique for obtaining a good estimator with a given sample size is the least squares (LS) approach. Nevertheless, the need to treat progressive sample sizes has become increasingly important. In fact, handling variable-sized series is necessary in many fields, such as signal processing, control, and instantaneous applications. An elegant approach to recursive identification is to derive it from off-line estimation, so that the LS method can be carried out recursively. For more details, see [13]. The current estimator is derived by incorporating new data into the previous estimator, which allows for continuous refinement and potential improvements in accuracy. The main tool for obtaining the RLS algorithm is the matrix inversion lemma, which leads to a formula without matrix inversion and offers excellent performance in terms of computation and memory space.

The RLS method has been successfully applied to many time series models; one of them is the EXPAR model (see [27], [28]). This model was introduced by [21] and can exhibit nonlinear phenomena such as limit cycles, jump phenomena, and non-normality. Due to the nonlinearity of the model, the obtained RLS algorithm was nonconventional, and nonlinear optimization was used. Specifically, in the context of real-time estimation, [24] suggested a direct estimation from the data for the nonlinear parameter. [19] utilized this method to model the vibrations and disturbances taking place during drilling.

The Restricted EXPAR model is therefore derived. It is worth noting that the restricted model is linear in the unknown parameters but still nonlinear with respect to the variable, thereby retaining all of its nonlinear behavior. The approach will perform well if the nonlinear parameter of the model is known from earlier research or can be readily determined in the context of real-time data.

In this chapter, we present an online recursive least squares algorithm for the estimation of the restricted EXPAR model. This approach is based on the matrix inversion lemma, which avoids matrix inversion and is efficient in terms of real-time computation and small memory requirements.

3.2 Restricted EXPAR(p) Process

The restricted exponential autoregressive EXPAR(p) process is given by the formula

$$Z_t = \sum_{j=1}^{p} \left(\vartheta_{1,j} + \vartheta_{2,j} \exp(-\gamma Z_{t-1}^2)\right) Z_{t-j} + \varepsilon_t, \qquad t \in \mathbb{Z}, \tag{3.1}$$

where $\{\varepsilon_t;\, t \in \mathbb{Z}\}$ is an i.i.d. sequence with distribution $\mathcal{N}(0, \sigma^2)$. The slope parameter, $\gamma > 0$, is assumed to be known. A heuristic determination of γ from data is given by

$$\hat{\gamma} = -\frac{\log(\varepsilon)}{\max_t Z_t^2}, \tag{3.2}$$

where ε is a small positive number (cf. [25]).

Let

$$\vartheta = (\vartheta_{1,1}, \vartheta_{2,1}, \ldots, \vartheta_{1,p}, \vartheta_{2,p})' \in \mathbb{R}^{2p}$$

be the vector of unknown parameters, and

$$\varphi(t) = \left(Z_{t-1},\, Z_{t-1} \exp(-\gamma Z_{t-1}^2),\, \ldots,\, Z_{t-p},\, Z_{t-p} \exp(-\gamma Z_{t-1}^2)\right)'.$$

Equation (3.1) can then be written in regression form:

$$Z_t = \vartheta' \varphi(t) + \varepsilon_t, \qquad t \in \mathbb{Z}. \tag{3.3}$$

Assumptions

A1: We suppose that the process is strictly stationary. If this assumption is violated, alternative methods such as using time-varying parameters or incorporating non-stationary components may be considered.

A2: The white noise sequence $\{\varepsilon_t; t \in \mathbb{Z}\}$ satisfies $\mathbb{E}(\varepsilon_t^4) < \infty$ for any $t \in \mathbb{Z}$, which implies $\mathbb{E}(Z_t^4) < \infty$. In case this assumption is violated, robust estimation techniques or methods requiring weaker moment conditions could be employed.

3.2.1 Least Squares Estimator

The least squares (LS) estimator minimizes the equation error with respect to ϑ:

$$W_n(\vartheta) = \frac{1}{n} \sum_{t=1}^{n} \big(Z_t - \vartheta' \varphi(t)\big)^2, \tag{3.4}$$

where n is the sample size. Since the criterion is quadratic in ϑ, the LS estimator is given by

$$\hat{\vartheta}_n = \left(\sum_{t=1}^{n} \varphi(t) \varphi(t)' \right)^{-1} \left(\sum_{t=1}^{n} \varphi(t) Z_t \right). \tag{3.5}$$

If ϑ denotes the true parameter vector, it can be shown that the LS estimator $\hat{\vartheta}_n$ converges asymptotically to ϑ and that

$$\sqrt{n}\,(\hat{\vartheta}_n - \vartheta) \;\; \xrightarrow{d} \;\; \mathcal{N}\big(0,\, \sigma^2 \Gamma^{-1}\big), \tag{3.6}$$

where $\Gamma = \mathbb{E}\big(\varphi(t)\varphi(t)'\big)$ and $\xrightarrow{d}$ denotes convergence in distribution.

3.3 Recursive Least Squares Estimation Algorithm

Let $Z_1, \dots, Z_t$ denote the available observations up to time t from the restricted EXPAR(p) model. The adopted optimality criterion in RLS estimation is the mean square error. Thus, the problem consists of finding the argument that minimizes

$$W_t(\vartheta) = \frac{1}{t}\sum_{k=1}^{t}\big(Z_k - \vartheta'\varphi(k)\big)^2. \tag{3.7}$$

The proposition below gives the RLS algorithm, which provides optimal estimators for the restricted EXPAR model.

Proposition 1. *The recursive least squares (RLS) algorithm for estimating the parameters of the EXPAR(p) model is given by the following system of recursive equations:*

$$\hat{\vartheta}_t = \hat{\vartheta}_{t-1} + \frac{P_{t-1}\varphi(t)}{\varphi(t)'P_{t-1}\varphi(t)+1}\big(Z_t - \hat{\vartheta}'_{t-1}\varphi(t)\big), \tag{3.8}$$

$$P_t = P_{t-1} - \frac{P_{t-1}\varphi(t)\varphi(t)'P_{t-1}}{\varphi(t)'P_{t-1}\varphi(t)+1}. \tag{3.9}$$

A standard choice of initial values is $P_0 = CI$ and $\vartheta_0 = 0$, where C is some large constant (e.g., $C = 10^6$), and I denotes the identity matrix.

Proof

From the objective function (3.7), the LS estimator $\hat{\vartheta}_t$ is given by

$$\left(\sum_{k=1}^{t}\phi(k)\phi(k)'\right)\hat{\vartheta}_t = \sum_{k=1}^{t}\phi(k)Z_k. \tag{3.10}$$

Denote

$$R_t = \sum_{k=1}^{t}\phi(k)\phi(k)', \tag{3.11}$$

it follows that

$$R_t = R_{t-1} + \phi(t)\phi(t)'. \tag{3.12}$$

After some straightforward modifications, one may derive the recursive equation using (3.1)– (3.5):

$$\hat{\vartheta}_t = \hat{\vartheta}_{t-1} + R_t^{-1}\phi(t)\big(Z_t - \hat{\vartheta}'_{t-1}\phi(t)\big). \tag{3.13}$$

The algorithm (3.13) is not suited for computing because we must invert a $p \times p$ matrix at each step. So we introduce $P_t = R_t^{-1}$. The matrix inversion lemma can be used to update P_t directly:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}, \tag{3.14}$$

see, for example, Ljung and Söderström, 1983, Lemma 2.1, p. 19. We obtain

$$P_t = P_{t-1} - \frac{P_{t-1}\phi(t)\phi(t)'P_{t-1}}{\phi(t)'P_{t-1}\phi(t) + 1}. \tag{3.15}$$

Hence the algorithm (3.8) state that the new estimate is equal to the previous estimate plus the prediction error multiplied by a gain.

The following proposition states the asymptotic properties of the RLS estimator.

Proposition 2. *Under assumptions A1 and A2:*

(i) $\hat{\vartheta}_t \to \vartheta$, *almost surely as* $t \to \infty$.

(ii) $\sqrt{t}(\hat{\vartheta}_t - \vartheta) \xrightarrow{d} \mathcal{N}(0, \sigma^2\Gamma^{-1})$.

Proof. The RLS algorithm needs an initial value to start up. The estimates resulting from (3.13) are then

$$\hat{\vartheta}_t = \big(P_0^{-1} + \sum_{k=1}^{t} \varphi(k)\varphi(k)'\big)^{-1}\big(P_0^{-1}\hat{\vartheta}_0 + \sum_{k=1}^{t} \varphi(k)Z_k\big). \tag{3.16}$$

For $P_0^{-1} \to 0$, the recursive estimates are asymptotically similar to

$$\hat{\vartheta}_t = \Big(\sum_{k=1}^{t} \varphi(k)\varphi(k)'\Big)^{-1}\Big(\sum_{k=1}^{t} \varphi(k)Z_k\Big), \tag{3.17}$$

that is, the recursive estimate becomes closer to the off-line estimate and has the same asymptotic properties.

Substituting Z_k by (3.3), we obtain

$$\hat{\vartheta}_t = \vartheta + \Big(\frac{1}{t}\sum_{k=1}^{t} \varphi(k)\varphi(k)'\Big)^{-1}\Big(\frac{1}{t}\sum_{k=1}^{t} \varphi(k)\varepsilon_k\Big). \tag{3.18}$$

From the ergodicity of Z_t and the independence between $\varphi(k)$ and ε_k, we have

$$\frac{1}{t}\sum_{k=1}^{t}\varphi(k)\varepsilon_k \to E(\varphi(k)\varepsilon_k) = 0,$$

thus (i) is verified.

For (ii), we have from (3.18):

$$\sqrt{t}(\hat{\vartheta}_t - \vartheta) = \Big(\frac{1}{t}\sum_{k=1}^{t}\varphi(k)\varphi(k)'\Big)^{-1}\Big(\frac{1}{\sqrt{t}}\sum_{k=1}^{t}\varphi(k)\varepsilon_k\Big).$$

The central limit theorem for martingale differences gives

$$\frac{1}{\sqrt{t}}\sum_{k=1}^{t}\varphi(k)\varepsilon_k \quad \to \quad \mathcal{N}(0, \sigma^2\Gamma),$$

leading to the result by applying Slutsky's theorem. □

3.4 Simulation Study

We simulate four restricted EXPAR(1) models with $\gamma = 1$ and $\sigma = 0.1$ for the first three models, and $\sigma = 1$ for the fourth model. The selection of parameters is based on ensuring the stationarity of the models. The parameters are as follows:

- **Model 1:** $\vartheta = (-0.9, -2)'$,
- **Model 2:** $\vartheta = (0.7, 1.5)'$,
- **Model 3:** $\vartheta = (0.8, 1)'$,
- **Model 4:** $\vartheta = (0.4, -0.9)'$.

We have added these additional models to provide a broader understanding of the estimator's performance under different parameter settings. The program has been written in R, and we have used the `RLS` function in the `MTS` package.

Figures 3.1–3.8 show the behavior of estimator means and variances for Models 1 to 4. We can clearly see that the parameters are very well estimated by the RLS algorithm and are consistent: the means converge to the true parameters, and the variances converge towards zero.

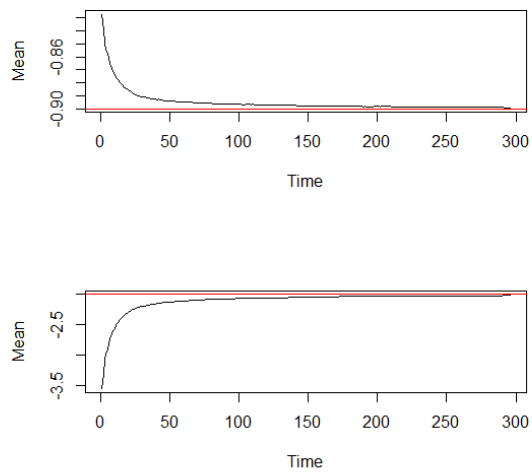


Figure 3.1: Mean of ϑ_t for Model 1.

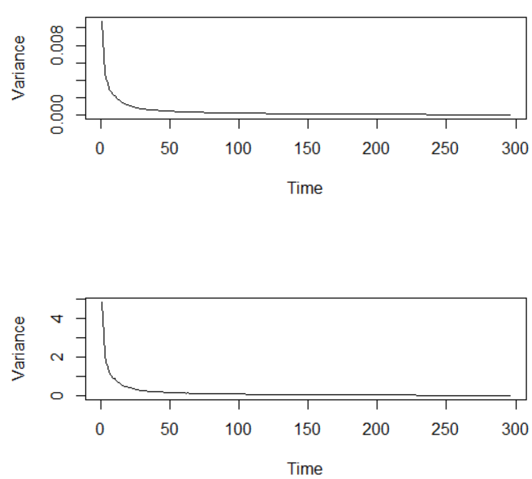


Figure 3.2: Variance of ϑ_t for Model 1.

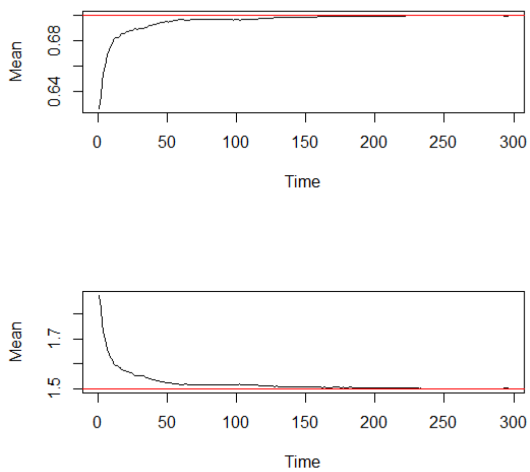


Figure 3.3: Mean of ϑ_t for Model 2.

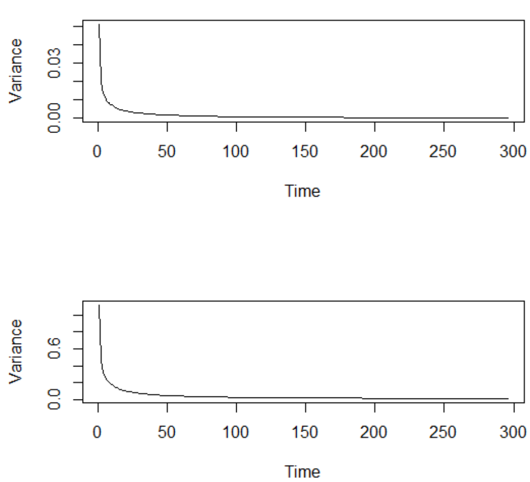


Figure 3.4: Variance of ϑ_t for Model 2.

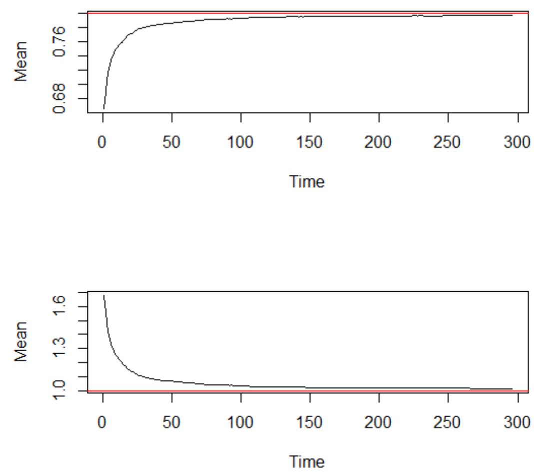


Figure 3.5: Mean of ϑ_t for Model 3.

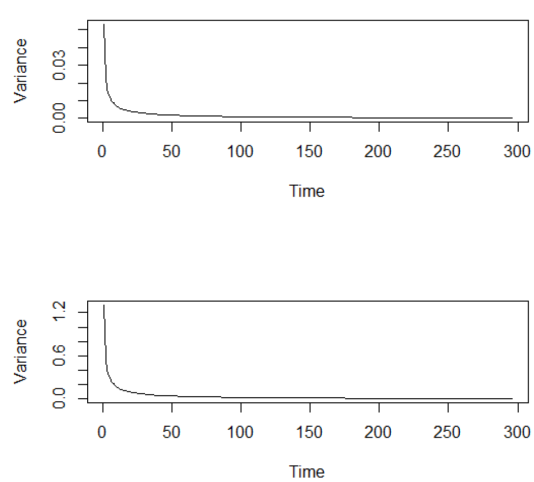


Figure 3.6: Variance of ϑ_t for Model 3.

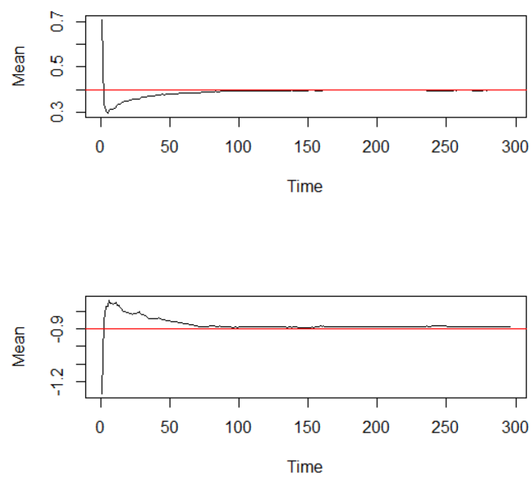


Figure 3.7: Mean of ϑ_t for Model 4.

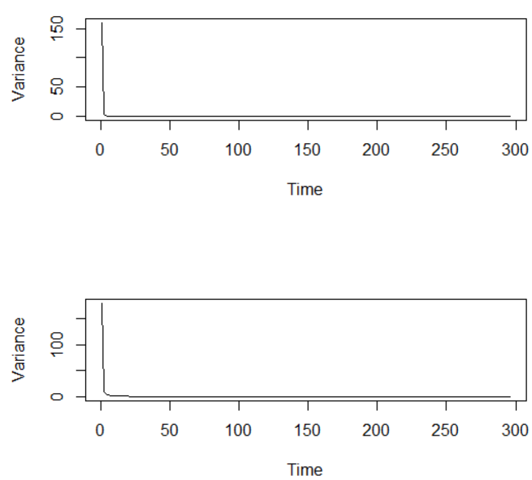


Figure 3.8: Variance of ϑ_t for Model 4.

Conclusion and Perspectives

In this thesis, we explored the restricted Periodic EXPAR(p) model, establishing the conditions for strict periodic stationarity and demonstrating its equivalence to the unrestricted model. Through the Quasi-Maximum Likelihood framework, we derived the theoretical properties of Likelihood Ratio and Lagrange Multiplier tests, leading to season-aggregated statistics tailored to the periodic setting.

Simulation studies revealed behavior consistent with the findings of [22] for the classical EXPAR model, notably the ability to capture complex and varied marginal density shapes through appropriate parameter selection. These results underscore the flexibility of the periodic EXPAR model in modeling nonlinear time series with seasonal features.

In an application to Algerian rainfall data, the restricted $\mathrm{PEXPAR}_{12}(1)$ model effectively captured seasonal variations, with patterns of volatility clustering evident in the variance fluctuations. This suggests that incorporating a periodic ARCH component into the PEXPAR framework could further enhance its ability to model heteroskedasticity and improve forecasting performance. Developing and evaluating such a periodic EXPAR–ARCH model represents a promising direction for future research.

Furthermore, we proposed Recursive Least Squares estimators for the restricted EXPAR model. These estimators are particularly suited for settings where the nonlinear parameter is known (e.g., from prior studies) or can be efficiently estimated in real-time applications. We established the asymptotic properties of these online estimators, and preliminary simulation results demonstrated their strong performance in practice. Exploring the integration of RLS algorithms with periodic structures and extending their applicability to real-world datasets presents another avenue for future research.

In summary, this work lays a solid foundation for advancing periodic nonlinear time series models, with future efforts focused on enhancing volatility modeling, developing real-time estimation techniques, and broadening applications to diverse domains such as finance, climate science, and biomedical data.

Appendix A

Proofs of the Nullity Tests for the Last Coefficient

Proof of Theorem 3

The LR statistic is given by

$$LR_m = -2\Big(L_n(\underline{\tilde{\phi}}, Z_1, \dots, Z_n) - L_n(\underline{\hat{\phi}}, Z_1, \dots, Z_n)\Big) = -2\Big(-\tfrac{m}{2}\sum_{i=1}^{S}\log(Q_{i,m}(\underline{\tilde{\phi}}_i)) + \tfrac{m}{2}\sum_{i=1}^{S}\log(Q_{i,m}(\underline{\hat{\phi}}_i))\Big)$$

$$= m\sum_{i=1}^{S}\log\left(\frac{Q_{i,m}(\underline{\tilde{\phi}}_i)}{Q_{i,m}(\underline{\hat{\phi}}_i)}\right).$$

As $m \to \infty$, LR_m asymptotically follows a χ^2 distribution with $2S$ degrees of freedom under H_0, thereby establishing the critical region for rejecting H_0 at level α.

Proof of Theorem 4

Under H_0, the estimator $\underline{\tilde{\phi}}_i$ is obtained by maximizing the log-likelihood $L_n(\underline{\phi})$ subject to $R\underline{\tilde{\phi}}_i = 0$, $\forall i$. Let the Lagrangian be defined as

$$L(\underline{\phi}, \lambda) = L_n(\underline{\phi}) + \lambda' \boldsymbol{R}\underline{\phi},$$

where λ is the $2S$-dimensional vector of multipliers, and $\boldsymbol{R}$ is the $2S \times 2pS$ block diagonal matrix defined as

$$\boldsymbol{R} = I_{S\times S} \otimes R_{2\times 2p} = \begin{pmatrix} R_{2\times 2p} & \cdots & 0_{2\times 2p} \\ \vdots & \ddots & \vdots \\ 0_{2\times 2p} & \cdots & R_{2\times 2p} \end{pmatrix}.$$

The restricted estimator $\underline{\tilde{\phi}}$ is then obtained by solving

$$G(\underline{\tilde{\phi}}) + \boldsymbol{R}'\tilde{\lambda} = 0.$$

The traditional LM test is given by

$$LM = \tilde{\lambda}'\boldsymbol{R}\tilde{\boldsymbol{\Gamma}}^{-1}\boldsymbol{R}'\tilde{\lambda} = G(\underline{\tilde{\phi}})'\tilde{\boldsymbol{\Gamma}}^{-1}G(\underline{\tilde{\phi}}) = \sum_{i=1}^{S} \tilde{\sigma}_i^2 G_i(\underline{\tilde{\phi}}_i)'\tilde{\Gamma}_i^{-1}G_i(\underline{\tilde{\phi}}_i) \ \sim \ \chi^2_{2S}.$$

The last equality follows because the matrix $\boldsymbol{\Gamma}$ is block diagonal. Let $\tilde{\varepsilon}_{S\tau+i}$ be the residual of the restricted estimation; therefore, the score vector evaluated at $\underline{\tilde{\phi}}_i$ is

$$G_i(\underline{\tilde{\phi}}_i) = \frac{1}{\tilde{\sigma}_i^2}(0, Z_i')\tilde{\boldsymbol{\varepsilon}}_i.$$

We can write $\tilde{\Gamma}_i$ as

$$\tilde{\Gamma}_i = \begin{pmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{pmatrix},$$

where

$$A_{i,11} = \begin{pmatrix} M_{i,1,1} & \cdots & M_{i,1,p-1} \\ \vdots & \ddots & \vdots \\ M_{i,p-1,1} & \cdots & M_{i,p-1,p-1} \end{pmatrix}_{2(p-1)\times 2(p-1)},$$

$$A_{i,12} = \begin{pmatrix} M_{i,1,p} \\ \vdots \\ M_{i,p-1,p} \end{pmatrix}_{2(p-1)\times 2}, \qquad A_{i,21} = \begin{pmatrix} M_{i,p,1} & \cdots & M_{i,p,p-1} \end{pmatrix}_{2\times 2(p-1)},$$

and the 2×2 matrix

$$A_{i,22} = M_{i,p,p}.$$

By the rules of partitioned matrices, the LM statistic becomes

$$LM = \sum_{i=1}^{S} \frac{1}{\tilde{\sigma}_i^2} \tilde{\varepsilon}_i' (0_{m \times 2(p-1)}, Z_i)\, \Gamma_i^{-1} \begin{pmatrix} 0_{2(p-1) \times m} \\ Z_i' \end{pmatrix} \tilde{\varepsilon}_i,$$

$$= \sum_{i=1}^{S} \frac{1}{\tilde{\sigma}_i^2} \tilde{\varepsilon}_i' Z_i \left(A_{i,22} - A_{i,21} A_{i,11}^{-1} A_{i,12} \right)^{-1} Z_i' \tilde{\varepsilon}_i.$$

This last form can be used in simulation.

Appendix B

Proofs for the Linearity Test

Proof of Theorem 5

Under the null hypothesis, the estimator is

$$\underline{\tilde{\phi}}_i = (K_{1.1})^{-1} K_{1.2},$$

where

$$K_{1.1} = \begin{bmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}^2 & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} \\ \vdots & \ddots & \vdots \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}^2 \end{bmatrix},$$

and

$$K_{1.2} = \begin{bmatrix} \sum_{\tau=1}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i} \\ \vdots \\ \sum_{\tau=1}^{m-1} Z_{S\tau+i-p} Z_{S\tau+i} \end{bmatrix}.$$

The corresponding residual sum of squares under the null hypothesis is

$$Q_{i,m}(\underline{\tilde{\phi}}_i) = \frac{1}{m} \sum_{\tau=0}^{m-1} \left(Z_{S\tau+i} - \sum_{j=1}^{p} \tilde{\phi}_{i,j} Z_{S\tau+i-j} \right)^2.$$

The likelihood ratio test statistic is then defined as

$$LR_m = m \sum_{i=1}^{S} \log \left(\frac{Q_{i,m}(\underline{\tilde{\phi}}_i)}{Q_{i,m}(\underline{\hat{\phi}}_i)} \right),$$

As $m \to \infty$, LR_m asymptotically follows a χ^2 distribution with Sp degrees of freedom under H_0.

Proof of Theorem 6

Under H_0, the estimator $\tilde{\vartheta}_i$ is obtained by maximizing the log-likelihood $L_n(\underline{\theta})$ subject to $R\vartheta_i = 0$. Consider the Lagrangian

$$L(\underline{\theta}, \lambda) = L_n(\underline{\theta}) + \lambda' \boldsymbol{R}\underline{\theta},$$

where λ is the pS-dimensional vector of multipliers and $\boldsymbol{R}$ is the $(pS \times 2pS)$ block diagonal matrix given as follows:

$$\boldsymbol{R} = I_{S\times S} \otimes R_{p\times 2p},$$

$$R = \begin{bmatrix} R_{p\times 2p} & \cdots & 0_{p\times 2p} \\ \vdots & \ddots & \vdots \\ 0_{p\times 2p} & \cdots & R_{p\times 2p} \end{bmatrix}.$$

The restricted estimator $\tilde{\underline{\theta}}$ is then derived by solving

$$G(\tilde{\underline{\theta}}) + \boldsymbol{R}'\tilde{\lambda} = 0.$$

The LM test is given by

$$LM = \tilde{\lambda}'\boldsymbol{R}\tilde{\boldsymbol{W}}^{-1}\boldsymbol{R}'\tilde{\lambda} = G(\tilde{\underline{\theta}})'\tilde{\boldsymbol{W}}^{-1}G(\tilde{\underline{\theta}}) = \sum_{i=1}^{S} \tilde{\sigma}_i^2 G_i(\tilde{\vartheta}_i)'(K\tilde{\Gamma}_i K')^{-1}G_i(\tilde{\vartheta}_i) \quad \sim \chi^2_{pS}.$$

Let $\tilde{\varepsilon}_{S\tau+i}$ be the residual of the restricted estimation, so the score vector evaluated at $\tilde{\vartheta}_i$ is

$$G_i(\tilde{\vartheta}_i) = \frac{1}{\tilde{\sigma}_i^2}(0, Z_i')\tilde{\varepsilon}_i.$$

Moreover,

$$K\tilde{\Gamma}_i K' = \begin{bmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{bmatrix},$$

where

$$A_{i,11} = \begin{bmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}^2 & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} \\ \vdots & \ddots & \vdots \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}^2 \end{bmatrix}_{p\times p},$$

$$A_{i,12} = A_{i,21} = \begin{bmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}^2 e^{-\gamma Z_{S\tau+i-1}^2} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} e^{-\gamma Z_{S\tau+i-1}^2} \\ \vdots & \ddots & \vdots \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} e^{-\gamma Z_{S\tau+i-1}^2} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}^2 e^{-\gamma Z_{S\tau+i-1}^2} \end{bmatrix}_{p\times p},$$

$$A_{i,22} = \begin{bmatrix} \sum_{\tau=0}^{m-1} Z_{S\tau+i-1}^2 e^{-2\gamma Z_{S\tau+i-1}^2} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} e^{-2\gamma Z_{S\tau+i-1}^2} \\ \vdots & \ddots & \vdots \\ \sum_{\tau=0}^{m-1} Z_{S\tau+i-1} Z_{S\tau+i-p} e^{-2\gamma Z_{S\tau+i-1}^2} & \cdots & \sum_{\tau=0}^{m-1} Z_{S\tau+i-p}^2 e^{-2\gamma Z_{S\tau+i-1}^2} \end{bmatrix}_{p\times p}.$$

By the rules of partitioned matrices, the LM statistic can then be written as

$$\begin{aligned} LM &= \sum_{i=1}^{S} \frac{1}{\tilde{\sigma}_i^2} \tilde{\varepsilon}_i' (0_{m\times p}, Z_i) \begin{bmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{bmatrix}^{-1} \begin{bmatrix} 0_{p\times m} \\ Z_i' \end{bmatrix} \tilde{\varepsilon}_i, \\ &= \sum_{i=1}^{S} \frac{1}{\tilde{\sigma}_i^2} \tilde{\varepsilon}_i' Z_i \left(A_{i,22} - A_{i,21} A_{i,11}^{-1} A_{i,12} \right)^{-1} Z_i' \tilde{\varepsilon}_i. \end{aligned}$$

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