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THESIS

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Existence and stability of solutions of certain neutral-type differential equations

Stream
Mathematics

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By
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Présentée en vue de l'obtention du diplôme de Doctorat en sciences

Existence et stabilité de solutions de certaines équations différentielles de type neutre

Filière
Mathématiques

Spécialité
Mathématiques Appliquées

Par
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Existence and stability of solutions of certain neutral-type differential equations

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Department of Mathematics

2025/2026

Supervisor: *Prof. Ahcene Djoudi*

DEDICATION

اهدي هذا العمل الى والديا، الى امي الرؤوم **عائشة** رحمها الله، نبع الحنان.... والى روح جدتي الفاضلة **هناء** رحمها الله وكذلك الى روح عمي الأستاذ الدكتور **نجيب** رحمه الله؛ الذي غادرنا وهو صغير، الى زوجتي العزيزة **ليلى** التي دعمتني وشجعتني كثيرا، وأبنائي **عائشة**، و**قدس**، و**أسامة**، والى عمتي **عقيلة** التي درستني وكفلتني في المراحل الاولى من تعليمي، الى جميع أفراد العائلتين الكبيرتين **بوحنيك** و**هرباش**.

I dedicate this work to my parents, to my compassionate mother **Aïcha**, may God have mercy on her, the source of tenderness... and to the soul of my virtuous grandmother **Hana**, may God have mercy on her, as well as to the soul of my uncle, Professor **Nadjib**, may God have mercy on him, who left us when he was young, to my dear wife **Leila**, who supported and encouraged me a lot, and my children **Aïcha**, **kouds**, and **Ousama**, and to my aunt **Akila**, who taught me and supported me in the early stages of my education, to all members of my large family, **Bouhnik** and **Herbache**.

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ملخــــص

الهدف من هــذه الأطروحة هو دراسة وجود واستقرار الحلول الدوريـة لبعض المعادلات وجمل المعـادلات التفـاضليـة الحيـادية ذات التأخـيرات الزمنيـة والمعامـلات المتغيرة. باستخـدام نظـرية النقطـة الثابتـة لكراسنوسيلسكي، نقـــدم مجموعــة من الشروط الكافيـة التي تضمـــن وجود الحلـول الدوريـة. يتضمن ذلك تحويل الجملة إلى صيغة تكاملية مكافئة قبل تطبيق الحلول المصفوفية الأساسية بالتوازي مع نظرية فلوكيه. بالإضـافة إلى ذلك، سنقـوم بتحليل الاستقـرار التقاربي لهذه الحلول، مما يؤدي إلى وضع شروط جديدة يمكنها ضمان الاستقـرار. يتم دعم الأهمية العملية لنتائجنا النظـرية من خلال أمثلة عدديــة، حيث يتم التحقق من صحة النهج المقترح وإبراز ملائمته في مجالات مختلفة مثل الدوائر الكهربائية ومســائل خطوط نقل الطاقة والإشارة وأنظمة التحكم والنمذجة البيولوجية. تمتد هذه الدراســة لتشمل أعمالًا سابقة، مقدمةً بذلك إطارًا تفصيليًا مخصصًا لدراسة جمل المعادلات التفاضلية الحيادية التي تحتوي على تأخيرات زمنية.

الكلمات المفتاحية: المعادلات والأنظمة التفاضلية الحيادية، التأخيرات الزمنية، النقطة الثابتة، نظرية فلوكيه، الحلول المصفوفية الأساسية، كراسنوسيلسكي، الحلول الدورية، الاستقرار التقاربي.

.

Abstract

The objective of this thesis is to study the existence and stability of periodic solutions for certain neutral differential equations and systems with time delays and variable coefficients. Using Krasnoselskii's fixed point theorem, we present a set of sufficient conditions that guarantee the existence of periodic solutions. This involves transforming the system into an equivalent integral form before applying fundamental matrix solutions in parallel with Floquet theory. In addition, we analyze the asymptotic stability of these solutions, leading to new conditions that can ensure stability. The practical significance of our theoretical results is supported by numerical examples, which verify the validity of the proposed approach and highlight its applicability in various fields such as electrical circuits, power transmission and signal propagation problems, control systems, and biological modeling. This study extends previous work by providing a detailed framework tailored to the study of neutral differential systems with time delays.

Keywords: Neutral differential equations and systems, Time delays, Fixed point, Floquet theory, Fundamental matrix solutions, Krasnoselskii, Periodic solutions, Asymptotic stability

Résumé

L'objectif de cette thèse est d'étudier l'existence et la stabilité des solutions périodiques de certaines équations et systèmes d'équations différentielles neutres avec retards temporels et coefficients variables. En utilisant le théorème du point fixe de Krasnoselskii, nous présentons un ensemble de conditions suffisantes garantissant l'existence de solutions périodiques. Cela implique la transformation du système en une forme intégrale équivalente avant d'appliquer les solutions matricielles fondamentales en parallèle avec la théorie de Floquet. De plus, nous analysons la stabilité asymptotique de ces solutions, ce qui conduit à permet de déduire de nouvelles conditions assurant la stabilité. L'importance pratique de nos résultats théoriques est appuyée par des exemples numériques, qui vérifient la validité de l'approche proposée et soulignent son adéquation dans divers domaines tels que les circuits électriques, les problèmes de transmission d'énergie et de propagation de signaux, les systèmes de commande et la modélisation biologique. Cette étude étend les travaux précédents en proposant un cadre détaillé spécialement conçu pour l'étude des systèmes d'équations différentielles neutres avec retards temporels.

Mots clés : Équations et systèmes différentiels neutres**,** Retards temporels, Poin fixe**,** Théorie de Floquet**,** Solutions matricielles fondamentales, Krasnoselskii, Solutions périodiques, Stabilité asymptotique.

Contents

Introduction

Neutral functional differential equations have emerged as powerful mathematical tools for modeling complex phenomena across science and engineering, particularly in dynamical systems with time delays. These equations have garnered significant attention due to their wide-ranging applications in fields such as electrical circuit theory, control systems, biological modeling, and signal processing. Recent advancements in calculus have further enriched these models by incorporating non-local operators and memory effects, enabling more accurate representations of intricate systems.

Numerous studies have introduced various methods to analyze the stability as well as the existence of periodic solution, with fixed point theory being a frequently applied approach. Further details on the topic are provided in [2, 4, 5, 6, 7, 8, 16, 17, 18, 19, 20, 21, 24, 29, 30, 31, 32, 33, 34, 35] and the references therein.

Fixed point theory is a vital and active area of research with wide-ranging applications across various fields (see [16], [17], [18], [21], [40]). This theory focuses on studying the conditions that guarantee the existence of one or more fixed points for a self-map defined on a given set. It emerged alongside the development of classical mathematical analysis and gained increasing attention later due to the need to prove the existence of solutions for differential and integral equations, which made it initially a purely analytical theory. Fixed point theory is divided into three main branches: fixed point theory in metric spaces, fixed point theory in topology, and fixed point theory in discrete systems. Among the fundamental and classical results in these branches are Banach's Fixed Point Theorem, Schauder's Fixed Point Theorem, and Krasnoselskii's Fixed Point Theorem.

Historical Development of Fixed Point Theory: Fixed point theory constitutes one of the fundamental frameworks in mathematical analysis, playing a pivotal role in both theoretical and applied disciplines, particularly in the study of differential equations, integral equations, and dynamical systems. The field experienced a significant milestone in 1922, when the Polish mathematician **Stefan Banach** formulated his celebrated Fixed Point Theorem, commonly known as the Contraction Principle. This theorem asserts that any contraction mapping defined on a complete metric space possesses a unique fixed point, which can be approximated through a simple iterative process. The profound influence of this result laid the groundwork for modern approaches to existence and uniqueness problems in ordinary differential equations. Subsequently, around 1930 [39], the Polish-American mathematician **Jakob Schauder** extended Banach's principle to a broader setting by proving that continuous self-maps on compact and convex subsets of topological vector spaces admit at least one fixed point. This generalization, known as Schauder's Fixed Point Theorem, greatly expanded the applicability of the theory to infinite-dimensional settings and provided powerful tools for addressing nonlinear problems in functional analysis and partial differential equations. In 1955, the Soviet mathematician **Mark Krasnoselskii** introduced his distinguished hybrid fixed point theorem, which served as a bridge between the results of Banach and Schauder. This theorem states that if a mapping can be expressed as the sum of a contraction and a compact operator on a closed and convex subset of a Banach space, then the mapping has a fixed point. This result, now known as Krasnoselskii's Fixed Point Theorem, has proven to be of great importance in the study of nonlinear integral and differential equations, especially those involving mixed or hybrid operators.

Later in the twentieth century, specifically in 1998, the American researcher **T. A. Burton** expanded and generalized Krasnoselskii's theorem into a more flexible formulation known as the Krasnoselskii–Burton Theorem. This formulation was designed to address complex systems such as delay differential equations and other nonlinear dynamical models arising in fields like physics, engineering, and biology. This advancement significantly enriched the applications of fixed point theory, further reinforcing its role as a central tool in modern nonlinear analysis.

Calculus and Stability Analysis Calculus, in both its differential and integral forms, provides a fundamental and flexible mathematical framework with broad applications in the fields of science, engineering, and technology. Recent advancements in this area have led to the development of new methodologies for analyzing the stability of solutions in differential equations and dynamical systems [36, 37, 38]. In the context of contemporary research, effective stability criteria for nonlinear integro-differential equations have been developed through the integration of classical Lyapunov theory with fixed-point theorems, such as those of Banach and Krasnoselskii. Lyapunov theory offers powerful analytical tools for assessing the stability of nonlinear dynamical systems, while fixed point theorems are employed to establish the existence and uniqueness of solutions.

This synergy between the two approaches has enabled the formulation of more comprehensive results that address issues related to time delays, nonlinearity, and integral dependencies in complex mathematical models. Numerous recent studies highlight the success of this combined framework in diverse applications, including neutral systems, delay differential equations, and biological models. This research direction has significantly extended previous results on neutral differential equations with time-varying delays, tackling previously unresolved challenges and enhancing the general applicability of stability analysis in more realistic and practical contexts.

Recent Developments in Mathematical Modeling Alongside theoretical contributions, recent years have witnessed significant advances in the modeling and simulation of dynamical systems. Notable developments include:

- The development of advanced numerical techniques for simulating and analyzing dynamical systems in electrical circuits and control theory, which has led to improved design, performance prediction, and stability verification in engineering systems.

- The application of differential equations in epidemiological modeling and biological systems, enabling researchers to more accurately simulate disease transmission dynamics, population behavior, and ecosystem interactions.

- Novel approaches to neutral differential systems using Floquet theory and matrix solutions

These contributions collectively reflect the growing sophistication of mathematical tools and their pivotal .

Recent Developments in Neutral Differential Systems with Time-Delays: Neutral differential systems with time varying delays have attracted increasing attention in the scientific community due to their structural complexity and their significant relevance in modeling various applied phenomena in fields such as engineering and biology. Djoudi and Ardjouni, together with their collaborators, are among the most prominent researchers who have made substantial contributions to this area, introducing important advances at both the analytical and numerical levels aimed at investigating the stability of such systems under nonlinear and complex delay conditions.

Djoudi and Ardjouni research [4, 5, 6, 7] focuses on integrating robust mathematical tools most notably Lyapunov's theory for stability analysis and fixed point theorems such as those of Banach and Krasnoselskii. This integration has enabled the construction of a comprehensive theoretical framework for proving the existence and uniqueness of solutions. The approach is particularly flexible in addressing systems that involve non constant delays and memory effects, thereby enhancing the precision of modeling and the analysis of long-term dynamical behavior.

Moreover, Djoudi, Ardjouni and his team [20, 31, 32, 33, 34, 35] have developed advanced numerical techniques that improve the efficiency of such analyses. These methods have been applied to realistic systems in areas including control theory, epidemic modeling, and biological systems, affirming the essential role of specialized mathematical tools in dealing with models characterized by complex and nonlinear dynamics.

These contributions represent a pivotal step toward expanding both the theoretical and practical understanding of neutral differential systems with time-delays and reinforcing the capacity to design more stable and reliable models for real-world applications.

In 2021, Boulaaras et al. published a comprehensive analysis of fractional-order neural networks, providing new insights into the stability and synchronization of these complex systems [12]. This work has opened new avenues for research in artificial intelligence and machine learning, particularly in the development of more robust and adaptable neural network architectures.

The current work extends these foundations by investigating periodic solutions in generalized neutral systems through Krasnoselskii's fixed-point theory. Our approach builds upon recent stability criteria for Caputo-derived systems while introducing new conditions for asymptotic stability in time-varying delayed systems. The subsequent analysis demonstrates how these theoretical advancements enable more robust modeling of transmission line phenomena and nonlinear circuit dynamics.

In 2010, Ding and Li [19] analyzed the below equation:

$$\begin{aligned}&\frac{d}{d\varkappa}u(\varkappa) - r\frac{d}{d\varkappa}u(\varkappa-\varsigma)\\&= q(\varkappa) - au(\varkappa) - aru(\varkappa-\varsigma) - c\Gamma(u(\varkappa)) + cr\Gamma(u(\varkappa-\varsigma)).\end{aligned}$$

Their work was motivated by issues in signal transmission lines and energy-related problems in electrical circuit models. They used Krasnoselskii's theory to define conditions for asymptotic stability as well as the existence of periodic solution, correcting a previous result by Angelov in 2007 [2].

Mansouri, Ardjouni, and Djoudi [31] expanded on this research in 2017 by investigating the following:

$$\frac{d}{d\varkappa}u(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\pounds(u(\varkappa - \varsigma(\varkappa)))$$
$$= q(\varkappa) - a(\varkappa)u(\varkappa) - a(\varkappa)r(\varkappa)\pounds(u(\varkappa - \varsigma(\varkappa))) - c(\varkappa)\Gamma(u(\varkappa)) + c(\varkappa)r(\varkappa)\Gamma(u(\varkappa - \varsigma(\varkappa))).$$

They used Krasnoselskii's theory to demonstrate asymptotic stability as well as the existence of periodic solution when coefficients and delay functions vary with time, a significant step towards more realistic modeling of physical systems.

In 2020, Guerfi and Ardjouni [20] examined a neutral differential system with constant delay:

$$\frac{d}{d\varkappa}u(\varkappa) - r\frac{d}{d\varkappa}u(\varkappa - \varsigma)$$
$$= \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa) + \mathcal{N}(\varkappa)ru(\varkappa - \varsigma) - c\Gamma(u(\varkappa)) + cr\Gamma(u(\varkappa - \varsigma)).$$

By using the fundamental matrix solution and Floquet theory, the authors converted the differential system into an integral system, making it suitable for the application of Krasnoselskii's theory. This approach provided a more comprehensive framework for analyzing periodic solutions in neutral systems.

Here, see [10] we analyze the following general neutral differential system, focusing on the asymptotic stability with the existence of its periodic solution:

$$\frac{d}{d\varkappa}u(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\pounds(u(\varkappa - \varsigma(\varkappa))) = \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa)$$
$$+\mathcal{N}(\varkappa)r(\varkappa)\pounds(u(\varkappa - \varsigma(\varkappa))) - c(\varkappa)\Gamma(u(\varkappa)) + c(\varkappa)r(\varkappa)\Gamma(u(\varkappa - \varsigma(\varkappa))),$$

here ς is a positive differentiable function, c and r are continuously and twic continuously differentiable. Moreover, the non-singular matrix i.e., $n \times n$ regard to continuous real value function is denoted by $\mathcal{N}$,$\mathcal{Q}$, $\pounds$, and Γ are assumed to be continuously differentiable functions.

The Krasnoselskii's fixed point theory is used on this system to demonstrate the sufficient condition for the stability and existence of periodic solution. This work extends and generalizes the findings of Ding, Li, Mansouri, Ardjouni, Djoudi, and Guerfi, providing a more comprehensive framework for analyzing neutral differential systems with variable coefficients and delays.

The thesis consists of three chapters:

Chapter One: serves as an introduction to fixed point theory, functional differential equations, and stability theory, where we establish the terminology and notations used. This chapter is a survey aimed at reviewing some fundamental definitions and theorems. It also presents some classical and recent results in the field of fixed point theory and functional differential equations, especially of the neutral type with time delays, and stability theory. We also present several examples and real-life models. We also talk about differential systems, and we review the role of fundamental matrix solutions and Floquet's theory in studying the existence and stability of these systems.

Chapter Two: This chapter is dedicated to presenting the most important research findings on the periodicity and stability of nonlinear neutral functional differential equations with time delays in Banach spaces. The desired results are based on Krasnoselskii's fixed point theorem. More specifically, we present and summarize very important results primarily concerning the existence and stability of periodic solutions for two problems that model issues arising from an electrical transmission line. The first problem was studied by Ding and Li [19], Using Krasnoselskii's fixed point theorem, we provide sufficient conditions for the existence of asymptotically periodic solutions. This study focuses on the challenges associated with analyzing integrated electronic circuits containing lossy transmission lines and nonlinear resistive loads with exponential voltage-current characteristics. The second problem, which we also discuss in this chapter, was studied by Mansouri, Ardjouni, and Djoudi [31]. This is considered an extension of the first problem, as it accounts for time-varying parameters, including time delays. Following the same approach, this study yields more comprehensive results that help stabilize oscillations along transmission lines.

Chapter Three: This chapter is of fundamental importance in this thesis as it contains significant results concerning the study of differential systems. These results are both novel and represent substantial advancements in the field, while simultaneously generalizing the results presented in Chapter 2. We note that these findings were published in a recent research article in early 2025, see [10] . Therefore, in this chapter we explore the existence of periodic solution by transforming the neutral differential system into an equivalent integral system using the fundamental matrix solution and Floquet theory. Also that we discusses the asymptotic-stability of equilibrium and periodic solution by applying Krasnoselskii's theorem to confirm the asymptotic-stability of trivial solution. Additionally, we present sufficient condition to ensure the stability of non-constant periodic solution. Practical examples are included to analyze the utility of the theoretical result, with particular emphasis on applications in electrical engineering and control systems.

The overall goal of this study is to further the understanding of transmission line problems, especially in how these problems relate to neutral differential equations and systems. As such, we provide new conditions intended to guarantee the stability and existence of periodic solution. These condition is more general and applicable to a wider range of systems than those previously established in the literature. It is anticipated that these findings will help to improve the understanding of dynamical systems with time delays, enhancing their applicability in a variety of scenarios across both science and engineering.

Our work builds upon the recent advancements in integral calculus and stability analysis, particularly those pioneered by Boulaaras and Ardjouni. By integrating these cutting-edge approaches with classical fixed-point theory, we aim to provide a more robust and versatile framework for analyze the complex dynamical system. This synthesis of modern and traditional methods offers new possibilities for modeling and predicting the behavior of system with memory effects, non-local interactions, and time-varying parameters.

The implications of this research extend beyond theoretical mathematics, offering potential applications in fields such as:

- Advanced control systems for renewable energy integration
- Predictive modeling in epidemiology and population dynamics
- Design of robust communication networks with variable delays
- Analysis of complex biological systems with memory effects
- Development of more efficient and stable power transmission systems

By providing a more comprehensive understanding of periodic solutions in neutral differential systems, this study aims to bridge the gap between theoretical advancements and practical applications, paving the way for innovative solutions to complex engineering and scientific challenges.

CHAPTER 1

Preliminaries and fundamental concepts

1.1 Background Concepts in Functional Analysis

Definition 1.1. *A metric d on a set X is a function $d : X \times X \to [0, \infty)$ such that for all x, y, $z \in X$*

i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

ii) $d(x, y) = d(y, x)$ (symmetry),

iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality). A metric space (X, d) is a set X with a metric d defined on X.

The metric space (X, d) is complete if every Cauchy sequence in X has a limit in X, i.e., every Cauchy sequence is convergent. We say that a sequence $(x_n)_{n \geq 0} \subseteq X$ is a Cauchy sequence if for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n,\ m > N$, $d(x_n, x_m) \leq \epsilon$.

Definition 1.2. *A linear space $(E, +, .)$ is a normed space if for each $x \in E$ there is a nonnegative real number $\|x\|$, called the norm of x, such that*

i) $\|x\| = 0$ if and only if $x = 0$,

ii) $\|\alpha x\| = |\alpha| \|x\|$ for each $\alpha \in \mathbb{R}$,

iii) $\|x + y\| \leq \|x\| + \|y\|$.

***Example* 1.1.** *Let $E = \mathbb{R}^n$, $n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:*

i) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, *for all* $x = (x_1, ..., x_n) \in \mathbb{R}^n$,

ii) $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, *for all* $x = (x_1, ..., x_n) \in \mathbb{R}^n$ *and* $p \in (1, \infty)$,

iii) $\|x\|_\infty = \max_{i=1,n} |x_i|$, *for all* $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

A norm induces a metric on the vector space, by $d(x, y) = \|x - y\|$.

Definition 1.3. *A Banach space is a complete normed space.*

***Example* 1.2.** *The space* $C([a, b], \mathbb{R}^n)$ *consisting of all continuous functions* $f : [a, b] \to \mathbb{R}^n$ *is a vector space over the real. The number* $\|f\| = \max_{a \le t \le b} |f(t)|$, *where* $\|.\|$ *is the norm in* $\mathbb{R}^n$, *defines a norm making* $(C, \|.\|)$ *a Banach space.*

***Example* 1.3.** *Let* $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ *and* $\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$ *to represent the spaces of continuous differentiable and continuous function* $\vartheta : \mathbb{R} \to \mathbb{R}^n$. *For any* $0 < T$, *we define*

$$\mathcal{C}_T = \{\vartheta \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n) \mid \vartheta(\varkappa + T) = \vartheta(\varkappa)\},$$

Under the supremum norm which forms a Banach space:

$$\|\vartheta\|_0 = \sup_{\varkappa \in \mathbb{R}} |\vartheta(\varkappa)| = \sup_{\varkappa \in [0,T]} |\vartheta(\varkappa)|,$$

For $x \in \mathbb{R}^n$, *in which* $|.|$ *indicates the infinite-norm. Furthermore, we denote* $\mathcal{C}_T \cap \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ *as* $\mathcal{C}_T^1$, *which constitutes a Banach space equipped with the given norm*

$$\|\vartheta\|_1 = \|\vartheta\|_0 + \|\vartheta'\|_0,$$

***Example* 1.4.** *Let* $\psi : [a, b] \to \mathbb{R}^n$ *a continuous function and let* S *be the set of functions continuous and bounded* $f : [a, \infty) \to \mathbb{R}^n$ *with* $f(t) = \psi(t)$ *for* $a \le t \le b$, *for* f, $g \in S$, *we defines*

$$d(f, g) = \|f - g\| = \sup_{a \le t < \infty} |f(t) - g(t)|$$

So (S, d) *is a complete metric space. Define*

$$M = \{\varphi : [0, \infty) \to \mathbb{R} / \ \varphi \in \mathbb{C}, \ |\varphi| \le 1, \ \varphi(t) \to 0 \text{ when } t \to \infty, \}$$

and

$$Q = \{\varphi : [0, \infty) \to \mathbb{R} / \ \varphi \in \mathbb{C}, \ |\varphi| \le 1\}.$$

Let $|.|$ *the supremum norm and let* $|.|_h$ *a norm weight defined by the data a function*

$h : [0, \infty) \to [1, \infty)$, $h(0) = 1$, $h(t) \to \infty$, *and for* $\varphi \in M$ *or* Q, *we pose*

$$|\varphi|_h = \sup_{t \ge 0} |\varphi(t)| / |h(t)|.$$

Example 1.5. $(M, \|.\|)$ *is a Banach space.*

Example 1.6. $(Q, |.|_h)$ *is a Banach space.*

Proof. Suppose that $\{\phi_n\}$ is from Cauchy in this space. The restriction of $\{\phi_n\}$ at the interval $[0, k]$ remains of Cauchy and therefore admits a continuous limit defined on this last interval. But this is true for everything $k = 1, 2, ...$ so we get a continuous limit on $[0, \infty)$ which clearly belongs to Q. □

Definition 1.4. *A subset M of a metric space (X, d) is compact if any sequence$\{x_n\}$ of M admits a subsequence with limit in M. M is relatively compact if every sequence its closure is compact, (i.e. $\overline{M}$ is compact).*

Definition 1.5. *Let U be an interval of $\mathbb{R}$ and let $\{f_n\}$ a series of functions with $f_n : U \to \mathbb{R}^p$. Let $\|.\|$ be any norm $\mathbb{R}^p$.*

(a) $\{f_n\}$ is uniformly bounded on U if there exists a $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in U$.

(b) $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that t_1, $t_2 \in U$ and $|t_1 - t_2| \leq \delta$ imply $|f_n(t_1) - f_n(t_2)| \leq \epsilon$, for all n.

Theorem 1.1 (Ascoli-Arzela [17]). *If $\{f_n\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

Example 1.7. *Consider the Banach space $C([a, b], \mathbb{R}^n)$ with the norm of supremum $\|f\| = \max_{a\leq t\leq b} |f(t)|$, with $|.|$ a norm of $\mathbb{R}^n$. Given two constant positive α and β, the set*

$$L = \{f \in C([a, b], \mathbb{R}^n) \;/\; \|f\| \leq \alpha, \;\; |f(u) - f(\nu)| \leq \beta |u - \nu|\},$$

is compact. This is a consequence of Ascoli's Theorem.

Example 1.8. *(a) Let (E, d) denote the space of bounded continuous functions $f : (-\infty, 0] \to \mathbb{R}^n$ with $d(\phi, \psi) = \sup_{-\infty<s\leq 0} |\phi(s) - \psi(s)|$ where $|.|$ is the Euclidean norm in $\mathbb{R}^n$*

(b) Then (E, d) is a Banach space.

(c) The set

$$L = \{f \in E/ \;\|f\| \leq 1, \;\; |f(u) - f(\nu)| \leq |u - \nu|,\}$$

is not compact in (E, d). To see this, consider the sequence of functions $\{f_n\}$ from $(-\infty, 0]$ into $[0, 1]$ with $f_n = 0$ for $t \leq -n$, f_n is the straight line between the points $(-n, 0)$ and $(0, 1)$. Any subsequence of $\{f_n\}$ converges pointwise to $f_n = 1$. But $d(f_n, 1) = 1$ for all n. Thus, there is no subsequence of $\{f_n\}$ with a limit in (E, d).

1.2 Fixed point theorems

Definition 1.6. *Let f be a mapping in the set M. we call fixed point of f any point x satisfying $f(x) = x$. If there exists such x, we say that f has a fixed point, which is equivalent to saying that the equation $f(x) - x = 0$ has a null solution.*

Definition 1.7. *Let (E, d) be a complete metric space. The operator $B : E \to E$ is called the contraction operator, if there exists a constant $0 \le \alpha < 1$ such that*

$$\forall x, y \in X, \ d(Bx, By) \le \alpha d(x, y).$$

Theorem 1.2. *[**Contraction Mapping Principle** [17]] Let (E, d) a complete metric space and let $B : E \to E$ a contraction mapping. Then there is one and only one point $x \in E$ with $Bx = x$. Furthermore, if $y \in E$ and if $\{y_n\}$ is defined inductively by $y_1 = By$, $y_{n+1} = By_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Bx = x$ has one and only one solution.*

Proof. Let $x_0 \in E$ and define a sequence $\{x_n\}$ in E by $x_1 = fx_0$, $x_2 = fx_1 = f^2x_0$,...,$x_n = fx_{n-1} = f^n x_0$. To see that $\{x_n\}$ is a Cauchy sequence, note that if $m > n$ then

$$\begin{aligned} d(x_n, x_m) &= d(f^n x_0, f^m x_0) \le \alpha d\left(f^{n-1}x_0, f^{m-1}x_0\right) \le ... \le \alpha^n d(x_0, x_{m-n}) \\ &\le \alpha^n \{d(x_0, x_1) + d(x_1, x_2) + ... + d(x_{m-n-1}, x_{m-n})\} \\ &\le \alpha^n \left\{d(x_0, x_1) + \alpha d(x_0, x_1) + ... + \alpha^{m-n-1} d(x_0, x_1)\right\} \\ &= \alpha^n d(x_0, x_1)\left[1 + \alpha + ... + \alpha^{m-n-1}\right] \\ &\le \frac{\alpha^n}{1-\alpha} d(x_0, x_1). \end{aligned}$$

Because $\alpha < 1$, the right side tends to zero as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence and (E, d) is complete so it has a limit $x \in E$. Now f is certainly continuous so

$$fx = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (fx_n) = \lim_{n\to\infty} x_{n+1} = x,$$

and x is a fixed point. To see that x is the unique fixed point, let $fx = x$ and $fy = y$. Then

$$d(x, y) = d(fx, fy) \le \alpha d(x, y),$$

and, because $\alpha < 1$, we conclude that $d(x, y) = 0$ so that $x = y$. This completes the proof. □

Definition 1.8 (**completely Continuous mapping**)**.** *Let K be a subset of a Banach space X and $A : K \to X$ mapping. if A is and $A(K)$ is contained in a compact subset of X, then we say that A is a compact mapping (we also say that A is completely continuous).*

Theorem 1.3 (**Brouwer's Fixed Point Theorem** [15]). *Let B be closed ball in $\mathbb{R}^n$. Then any continuous mapping $f : B \to B$ has at least one fixed point.*

Theorem 1.4 (**Schauder's fixed point theorem**[39]). *Let K be a nonempty closed convex bounded subset of a Banach space $(X, \|.\|)$. Then every continuous compact mapping $f : K \to K$ has a fixed point.*

Proof. Let $f : K \to K$ be a continuous mapping. Since K is compact, f is uniformly continuous; Thus, given $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in K$, we have $\|f(x) - f(y)\| \leq \epsilon$, provided $\|x - y\| \leq \delta$. Furthermore, there exists a nite set of points $\{x_1, x_2, ..., x_n\} \subset K$ such that $K \subset \cup_{1\leq j\leq n} B(x_j, \delta)$ where $B(x_j, \delta)$ is the open ball of centre x_j and radius δ. The vector space $L = Vect(f(x_j))_{1\leq j\leq n}$, is finite dimensional, and therefore , and $K^* = K \cap L$ is non-empty, compact, convex and finite dimensional.
For $1 \leq j \leq n$, we define the continuous function $\psi_j : X \to \mathbb{R}$ by

$$\psi_j(x) = \{.0, \quad if \ \ \|x - x_j\| \geq \delta, 1 - \frac{\|x - x_j\|}{\delta}, \ otherwise,$$

we see that ψ_j is strictly positive on $B(x_j, \delta)$ and vanishes elsewhere. Therefore we have $\sum_{j=1}^{n} \psi_j(x) > 0$, for all $x \in K$, and this continuous function is therefore bounded below on K.
We can therefore define a partition of unity,

$$\varphi_j(x) = \frac{\psi_j(x)}{\sum_{k=1}^{n} \psi_k(x)},$$

which satisfy $\sum_{j=1}^{n} \varphi_j(x) = 1$, for all $x \in K$. We put then, for $x \in K$, $g(x) = \sum_{j=1}^{n} \varphi_j(x) f(x_j)$.
g is continuous (because it is the sum of the continuous functions) and its values are taken into K^* because $g(x)$ is a center of gravity of $f(x_j)$. So if we take the restriction $g_{|K^*} : K^* \to K^*$, (by Brouwer's theorem) g has a fixed point $y \in K^*$. Furthermore

$$f(y) - y = f(y) - g(y) = \sum_{j=1}^{n} \varphi_j(y) f(y) - \sum_{j=1}^{n} \varphi_j(y) f(x_j) = \sum_{j=1}^{n} \varphi_j(y) (f(y) - f(x_j)),$$

or if $\varphi_j(y) \neq 0$ then $\|y - x_j\| < \delta$, and so $\|f(y) - f(x_j)\| < \epsilon$. We have for every j, $\|\varphi_j(y)(f(y) - f(x_j))\| \leq \epsilon\varphi_j(y)$, and so

$$\|f(y) - y\| \leq \sum_{j=1}^{n} \|\varphi_j(y)(f(y) - f(x_j))\| \leq \sum_{j=1}^{n} \epsilon\varphi_j(y) = \epsilon.$$

So, for any integer m, we can find a point $y_m \in K$ so that $\|f(y_m) - y_m\| < 2^{-m}$. And since K is compact, from the sequence $(y_m)_{m\in\mathbb{Z}}$ we can extract a subsequence (y_{m_k}) which converges to a point $y^* \in K$. Then, f being continuous, the sequence $(f(y_{m_k}))$ converges to $f(y^*)$, and we conclude that $f(y^*) = y^*$, i.e. y^* is a fixed point of f on K. □

In 1955 **Krasnoselskii's ([17], [40])** observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then, **Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.** For better understanding this observation of Krasnoselskii, consider the following differential equation.

$$x'(t) = -a(t)\,x(t) - g(t,x)\,. \tag{1.1}$$

or $a(t+T) = a(t)$ and $g(t+T,x) = g(t,x)$ for a certain $T > 0$. We can transform this equation in another form while writing, formally

$$x'(t)\exp\left(\int_0^t a(s)\,ds\right) = -a(t)\,x(t)\exp\left(\int_0^t a(s)\,ds\right) - g(t,x)\exp\left(\int_0^t a(s)\,ds\right),$$

thus

$$x'(t)\exp\left(\int_0^t a(s)\,ds\right) + a(t)\,x(t)\exp\left(\int_0^t a(s)\,ds\right) = -g(t,x)\exp\left(\int_0^t a(s)\,ds\right),$$

or

$$\left(x(t)\exp\left(\int_0^t a(s)\,ds\right)\right)' = -g(t,x(t))\exp\left(\int_0^t a(s)\,ds\right),$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^t \left(x(u)\exp\left(\int_0^u a(s)\,ds\right)\right)' du = -\int_{t-T}^t g(u,x(u))\exp\left(\int_0^u a(s)\,ds\right) du,$$

what gives

$$x(t) = x(t-T)\exp\left(-\int_{T-t}^t a(s)\,ds\right) - \int_{t-T}^t g(u,x(u))\exp\left(-\int_u^t a(s)\,ds\right) du. \tag{1.2}$$

If we suppose that $\exp\left(-\int_{T-t}^t a(s)\,ds\right) = \alpha < 1$, and if $(X, \|.\|)$ is the Banach space of functions $\phi : \mathbb{R} \to \mathbb{R}$ continuous and T -periodic, then the equation (1.2) can be written as

$$\phi(t) = (B\phi)(t) + (A\phi)(t) = P\phi(t)\,,$$

where B is contraction provides that the constant $\alpha < 1$ and A is compact mapping. This example shows the birth of the mapping $P\phi(t) = B\phi(t) + A\phi(t)$ who is identified with a sum of a contraction and a compact mapping. Thus, the search of the solution for (1.2) requires an adequate theorem which applies to this hybrid operator P and who can conclude the existence for a fixed point which will be, in his turn, solution of the initial equation (1.1). Krasnoselskii found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result [16], [40].

Theorem 1.5 (**Krasnoselskii**). *Let D be a closed bounded convex nonempty subset of a Banach space $(X, \|.\|)$. Suppose that A and B map D into X such that*

i) A is compact and continuous,

ii) B is a contraction mapping,

iii) $x; y \in D$, implies $Ax + By \in D$,

Then there exists $z \in D$ with $z = Az + Bz$.

Proof. According to the condition (ii) we have

$$\begin{aligned} \|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\ &\leq \|x-y\|+\|Bx-By\| \\ &\leq \|x-y\|+\alpha\,\|x-y\| \\ &= (1+\alpha)\,\|x-y\|\,, \end{aligned}$$

and

$$\begin{aligned} \|(I-B)\,x-(I-B)\,y\| &= \|(x-y)-(Bx-By)\| \\ &\geq \|x-y\|-\|Bx-By\| \\ &\geq \|x-y\|-\alpha\,\|x-y\| \\ &\geq (1-\alpha)\,\|x-y\|\,. \end{aligned}$$

In short

$$(1-\alpha)\,\|x-y\| \leq \|(I-B)\,x-(I-B)\,y\| \leq (1+\alpha)\,\|x-y\|\,.$$

This inequality shows that $(I-B) : D \to (I-B)\,D$ is continuous and one to one. Thus, $(I-B)^{-1}$ exist and is continuous. Let us pose $U = (I-B)^{-1}A$. It is clear that U is compact mapping, because U is a composition of a continuous mapping with a compact. Under the theorem of Schauder, U has a fixed point, i.e.

$$\exists z \in D \quad such\ that \quad (I-B)^{-1}Az = z.$$

This is equivalent to $z = Az + Bz$. □

Remark 1.1. *If $A = 0$, the theorem become the theorem of Banach. If $B = 0$, then the theorem is not other than the theorem of Schauder.*

1.3 Delay differential equations

1.3.1 Fundamental concepts

Suppose $\tau > 0$ is a given real number, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-\tau, 0]$ we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element φ in C by $|\varphi| = \sup_{-\tau \leq s \leq 0} |\varphi(s)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If

$$t_0 \in \mathbb{R},\ A \geq 0 \text{ and } x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n),$$

then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$. For $\Omega \subseteq \mathbb{R} \times C$, and $f : \Omega \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t), \tag{1.3}$$

is a retarded functional differential equation on Ω and will denote this equation by DDE. The number τ is called the delay. The case $\tau = 0$ corresponds with an ordinary differential equation. It is clear that an appropriate initial condition at time $t = t_0$ must at least specify the vector x for all $t \in [t_0 - \tau, t_0]$, i.e

$$x(t) = \psi(t),\ t \in [t_0 - \tau, t_0]. \tag{1.4}$$

Here $\psi : [t_0 - \tau, t_0] \to \mathbb{R}^n$ is a known function, usually we suppose ψ to be a continuous function. The function ψ is called the initial function of the delay differential equation. Hence, the initial value problem of (1.3) is given by the following relation

$$\begin{cases} x'(t) = f(t, x_t),\ t \geq t_0, \\ x(t) = \psi(t),\ t_0 - \tau \leq t \leq t_0, \end{cases} \tag{1.5}$$

where ψ is a given function defined on $t \in [t_0 - \tau, t_0]$.

Definition 1.9. *Equation (1.3) is called*

(i) *linear if* $f(t,\psi) = L(t)\psi$*, where* $L(t)$ *is linear for each* t.

(ii) *nonhomogeneous if* $f(t,\psi) = L(t)\psi + h(t)$*, where* $h(t) \neq 0$.

(iii) *autonomous if* $f(t,\psi) = g(\psi)$*, where* g *does not depend on* t.

***Example* 1.9.** *The following equations are delay differential equations:*

$$x'(t) = x(t) + x(t-4), \tag{1.6}$$

$$x'(t) = a(t)x(t) + b(t)x'(t-\tau(t)) + h(t), \tag{1.7}$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)\,ds, \tag{1.8}$$

where $a(t)$, $b(t)$, $\tau(t)$ *are continuous functions. Equation (1.6) represents an autonomous linear differential equation with constant delay* $\tau = 4$*, equation (1.7) is a linear differential equation with non-homogeneous non-autonomous functional delay, and equation (1.8) represents a delay linear integro-differential equation.*

Definition 1.10. *A function* x *is said to be a solution of (1.3) on* $[t_0 - \tau, t_0 + A]$ *if there are* $t_0 \in \mathbb{R}$, $A > 0$ *such that* $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$ *and* x *satisfies (1.3) for* $t \in [t_0, t_0 + A]$*. In such a case, we say that* x *is a solution of (1.3) on* $[t_0 - \tau, t_0 + A]$*. For a given* $t_0 \in \mathbb{R}$ *and a given* $\psi \in C$*, we say that* $x = x(t, t_0, \psi)$ *is a solution of (1.5) with initial value at* t_0 *or simply a solution of (1.5) through* (t_0, ψ) *if there is an* $A > 0$ *such that* $x(t, t_0, \psi)$ *is a solution of (1.5) on* $[t_0 - \tau, t_0 + A]$ *and* $x_{t_0}(t_0, \psi) = \psi$.

Lemma 1.1. *[21] Let* $t_0 \in \mathbb{R}$, $\psi \in C$*, and* $f : \Omega \subset \mathbb{R} \times C \to \mathbb{R}^n$ *be a continuous function. Then* $x(t, t_0, \psi)$ *is a solution of (1.5) at* (t_0, ψ) *if and only if* $x(t, t_0, \psi)$ *is a solution of the integral equation:*

$$\begin{cases} x(t) = \psi(0) + \displaystyle\int_{t_0}^{t} f(u, x_u)\,du, \quad t \geq t_0, \\ x_{t_0} = \psi. \end{cases} \tag{1.9}$$

Proof. • **Necessary condition** Let $x(t, t_0, \psi)$ a solution of (1.5), then:

$$\begin{cases} x'(t) = f(t, x_t), \quad t \geq t_0, \\ x_{t_0} = \psi. \end{cases}$$

By integration, we get:

$$\int_{t_0}^{t} x'(u)\,du = x(t) - x(t_0) = \int_{t_0}^{t} f(u, x_u)\,du, \quad t \geq t_0.$$

Given that $x(t_0) = x(t_0 + 0) = x_{t_0}(0) = \psi(0)$, we obtain:

$$\begin{cases} x(t) = \psi(0) + \int_{t_0}^{t} f(u, x_u)\, du, \\ x_{t_0} = \psi. \end{cases}$$

• **Sufficient condition** Let $x(t, t_0, \psi)$ be a solution of the integral equation (1.9). Then:

$$x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(u, x_u)\, du.$$

Since $f(t, x_t)$ is continuous in t, by the Mean Value Theorem we have:

$$\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(u, x_u)\, du = f(t, x_t),$$

which completes the proof. □

Lemma 1.2. *[21] If $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n)$, then x_t is continuous in t for $t \in [t_0, t_0 + A]$.*

Proof. Since x is continuous on $[t_0 - \tau, t_0 + A]$, it is uniformly continuous. Thus, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $|x(t) - x(\tau)| < \epsilon$ when $|t - \tau| < \delta$. Consequently, for $t, \tau \in [t_0, t_0 + A]$ with $|t - \tau| < \delta$, we have:

$$|x(t + s) - x(\tau + s)| < \epsilon \quad \forall s \in [-\tau, 0],$$

which proves the continuity of x_t. □

Theorem 1.6. *[**Existence** [21]] Let Ω be an open subset of $\mathbb{R} \times C$ and $f(t, \psi)$ continuous on Ω. For any $(t_0, \psi) \in \Omega$, there exists a solution of (1.3) passing through (t_0, ψ).*

Definition 1.11. *The function $f(t, \psi)$ is Lipschitz in ψ on a compact set $K \subset \mathbb{R} \times C$ if there exists a constant $k > 0$ such that for all $(t, \psi_i) \in K$, $i = 1, 2$:*

$$|f(t, \psi_1) - f(t, \psi_2)| \leq k\|\psi_1 - \psi_2\|. \tag{1.10}$$

Theorem 1.7. *[**Existence and Uniqueness** [21]] Let $\Omega \subset \mathbb{R} \times C$ be open, $f : \Omega \to \mathbb{R}^n$ continuous, and $f(t, \psi)$ Lipschitz in ψ on each compact subset of Ω. Then for any $(t_0, \psi) \in \Omega$, there exists a unique solution of (1.3) through (t_0, ψ).*

1.3.2 Method of Steps

The method of steps is a classical analytical technique used to solve certain classes of delay differential equations (DDEs). Although this method is often regarded as being too tedious for practical use, the availability of computer algebra systems has significantly reduced the computational burden associated with its implementation (see [23]).

Consider the following linear delay differential equation with constant delays:

$$x'(t) = a_0 x(t) + a_1 x(t - w_1) + \cdots + a_m x(t - w_m), \tag{1.11}$$

where the initial function is prescribed as

$$x(t) = \phi(t), \qquad -\max(w_i) \leq t \leq 0.$$

Let

$$b = \min_{1 \leq i \leq m} w_i.$$

Then, for all $t \in [0, b]$, the delayed arguments $x(t - w_i)$ are completely determined by the initial function ϕ. Consequently, equation (1.11) reduces on the interval $[0, b]$ to an ordinary differential equation of the form

$$x'(t) = a_0 x(t) + a_1 \phi(t - w_1) + \cdots + a_m \phi(t - w_m).$$

Integrating, we obtain

$$x(t) = x(0) + \int_0^t \left(a_0 x(s) + a_1 \phi(s - w_1) + \cdots + a_m \phi(s - w_m)\right) ds, \qquad 0 \leq t \leq b.$$

Once the solution is known on $[0, b]$, the same procedure can be applied on the next interval $[b, 2b]$. In this case, the solution is given by

$$x(t) = x(b) + \int_b^t \left(a_0 x(s) + a_1 \phi(s - w_1) + \cdots + a_m \phi(s - w_m)\right) ds, \qquad b \leq t \leq 2b. \tag{1.12}$$

This iterative process can be continued step by step, provided that the resulting integrals remain tractable. Although the method may become cumbersome for general problems, it can be efficiently implemented for certain classes of equations using symbolic computation tools.

***Example* 1.10.** *As a simple illustration, consider the delay differential equation*

$$x'(t) = x(t - 7),$$

with the constant initial function

$$x(t) = 4, \qquad -7 \leq t \leq 0.$$

On the interval $[0, 7]$*, we have*

$$x'(t) = 4,$$

which yields

$$x(t) = 4t + 4, \qquad 0 \leq t \leq 7.$$

Using this result, the solution on the interval $[7, 14]$ *is obtained as*

$$x(t) = \int_7^t x(s-7)\,ds + x(7) = 2t^2 - 24t + 102.$$

This stepwise procedure can be easily implemented in computer algebra systems such as Maple using a simple `for` *loop.*

Example 1.11. *Consider the delay differential equation*

$$x'(t) = a\,x(t-\tau),$$

where a *and* τ *are constants with* $\tau > 0$*, together with the constant initial function*

$$\phi(t) = c, \qquad t_0 - \tau \leq t \leq t_0.$$

Applying the method of steps, the solution can be expressed in the closed form

$$x(t) = c \sum_{n=0}^{\left\lfloor \frac{t-t_0}{\tau} \right\rfloor + 1} \frac{a^n}{n!}\Big(t - t_0 - (n-1)\tau\Big)^n. \tag{1.3}$$

The method of steps can also be applied to delay differential equations with variable delays. Consider, for instance, the equation

$$x'(t) = f\Big(t, x(t), x(t-\tau(t))\Big), \tag{1.13}$$

where $\tau(t)$ is a continuous delay function. If there exists a constant $d > 0$ such that

$$\inf_{t \geq t_0} \tau(t) = d,$$

then the method of steps applies in a straightforward manner. In this case, the solution can be constructed iteratively on successive intervals of length d, where on each step equation (1.13) reduces to an ordinary differential equation of the form

$$x'(t) = f\Big(t, x(t), \psi_k(t-\tau(t))\Big),$$

with ψ_k denoting the known solution from the previous step.

1.3.3 Neutral delay differential equations

Definition 1.12. *[21] Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, φ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω. if D is continuous together with its first and second Fréchet derivatives with respect to φ and D_φ, the derivative with respect to φ, is atomic at β on Ω. Suppose that $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. Consider the neutral delay differential equation*

$$\frac{d}{dt} D(t, x_t) = f(t, x_t), \tag{1.14}$$

Definition 1.13. *[21] A function x is said to be a solution of (1.14) on $[t_0 - r, t_0 + A]$ if there are $t_0 \in \mathbb{R}$, $A > 0$ such that*

$$x \in C([t_0 - r, t_0 + A], \mathbb{R}^n), \ (t, x_t) \in \Omega, \ t \in [t_0, t_0 + A],$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (1.14) on $[t_0, t_0 + \sigma]$. For a given $t \in \mathbb{R}$, $\psi \in C$ and $(t_0, \psi) \in \Omega$ we say $x(t, t_0, \psi)$ is a solution of equation (1.14) with initial value ψ at t_0 or simply a solution through (t_0, ψ), if there is an $A > 0$ such that $x(t, t_0, \psi)$ is a solution of (1.14) on $[t_0 - r, t_0 + A]$ and $x_{t_0}(t_0, \psi) = \psi$; we say $x(t, t_0, \psi)$ is a solution of (1.14) on $[t_0 - r, \infty)$ if four every $A > 0$, $x(t, t_0, \psi)$ is a solution of equation (1.14) on $[t_0 - r, t_0 + A]$ et $x_{t_0}(t_0, \psi) = \psi$.

We now review the existence and uniqueness theory for the solution of the equation (1.14).

Theorem 1.8. *[**Existence** [21]] if Ω is an open set in $\mathbb{R} \times C$ and $(t_0, \psi) \in \Omega$, then there exists a solution of the NDDE (D, f) through (t_0, ψ).*

Theorem 1.9. *[**Uniqueness** [21]] if Ω is an open set in $\mathbb{R} \times C$ and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω, then, for any $(t_0, \psi) \in \Omega$ there exists a unique solution of the NDDE (D, f) through (t_0, ψ).*

We have the following example

***Example* 1.12.** *The following expressions*

$$\begin{aligned} x'(t) &= x(t-1) + \left[x'(t-3) + 1\right]^2, \\ x''(t) &= x\left(\frac{t}{2}\right) + x'(t-1) - x'(t-3). \end{aligned}$$

are neutral delay differential equations.

Remark 1.2. *There are many problems with time delay in the scientific literature, and some of them have attracted considerable attention, including the following: Economic model, Controlling a ship, Biology model, Mixing of Liquids, The sunflower equation, see [16]*

1.4 Stability theory for delay differential equations

Accurate Stability Classification Based on Observed Behavior

Graphical Representation

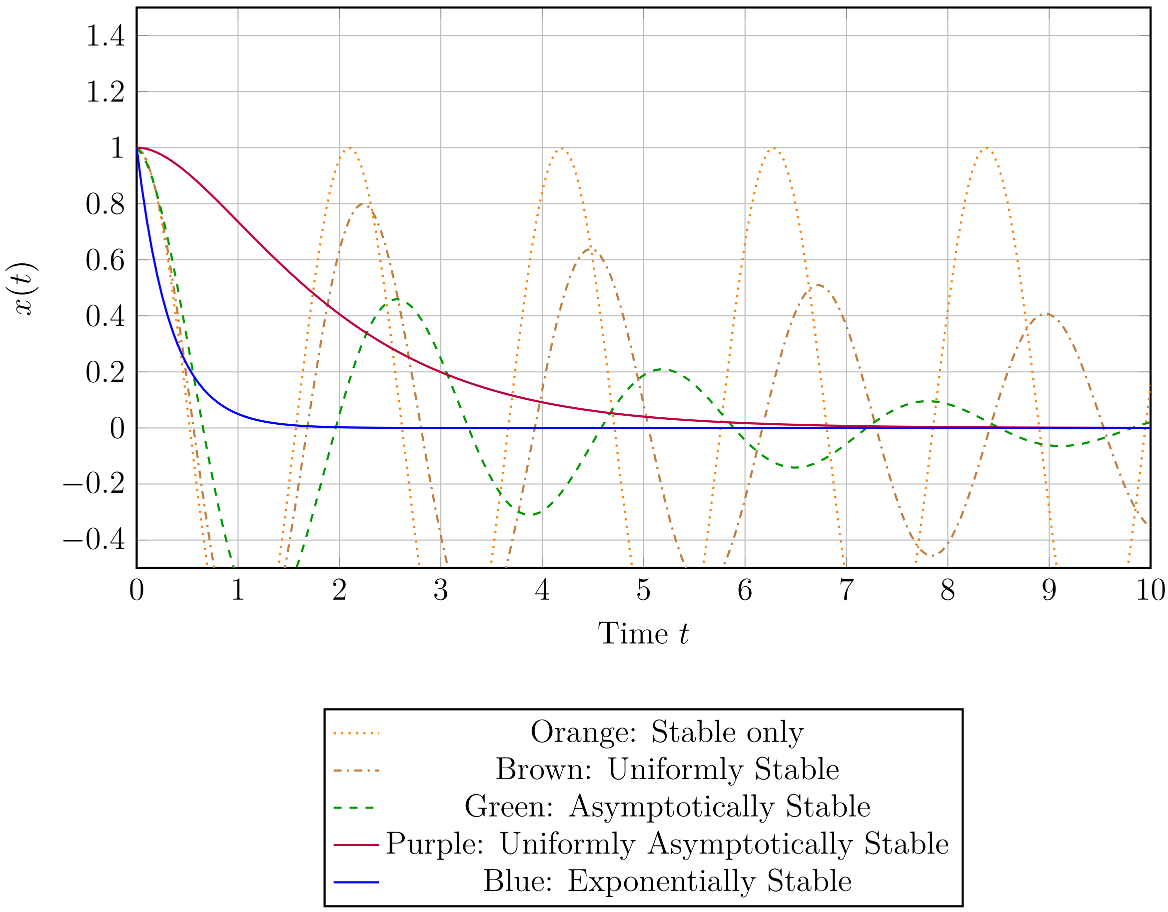


Figure 1.1: Stability classification matching the original plot exactly.

Suppose that $f : \mathbb{R} \times C \to \mathbb{R}^n$, is continuous and consider the delay differential equation

$$x'(t) = f(t, x_t). \tag{1.15}$$

The function f will be supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution $x(t, t_0, \psi)$ through (t_0, ψ) is continuous in (t, t_0, ψ) in the domain of definition of the function.

Definition 1.14. *[17] Suppose $f(t,0)=0$ for all $t\in\mathbb{R}$. The solution $x=0$ of equation (1.15) is said to be stable if for any $t_0\in\mathbb{R}$, $\epsilon>0$, there is a $\delta=\delta(\epsilon,t_0)>0$ such that $\|\psi\|\leq\delta$ implies $|x(t,t_0,\psi)|\leq\epsilon$ for $t\geq t_0$. The soluition $x=0$ of equation (1.15) is said to be uniformly stable if the number δ in definition is independent of t_0.*

Definition 1.15. *[17] The solution $x=0$ of equation (1.15) is said to be asymtotically stable if it is stable and there is a $b_0=b_0(t_0)$ such that $\|\psi\|\leq\delta$ implies that $x(t,t_0,\psi)\to 0$ as $t\to\infty$. The solution $x=0$ of equation (1.15) is said to be uniformly asymtotically stable if it is uniformly stable and ther is $b_0>0$ such that for every $\eta>0$ there is a $c_0(\eta)$ such that $\|\psi\|\leq b_0$ implies $|x(t,t_0,\psi)|\leq\eta$ for $t>t_0+c_0(\eta)$ for every $t\in\mathbb{R}$.*

Lyapunov's direct method has long been viewed as the main classical method of studying stability problems in many areas of differential equations. The difficulty of this method is to look for a suitable Lyapunov functional or Lyapunov function.

If $V:\mathbb{R}\times C\to\mathbb{R}$ is continuous and $x(t,t_0,\psi)$ is a solution of (1.15) through (t_0,ψ), we define

$$V'(t,\psi)=\lim_{h\to 0^+}\sup\frac{1}{h}\left[V\left(t+h,x_{t+h}(t,\psi)\right)-V(t,\psi)\right].$$

The function $V'(t,\psi)$ is the upper right-hand derivative of $V(t,\psi)$ along the solution of (1.15).

Theorem 1.10 ([21]). *Suppose $f:\mathbb{R}\times C\to\mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$, and u, ν, $w:\mathbb{R}^+\to\mathbb{R}^+$ are continuous non-decreasing functions, $u(s)$ and $\nu(s)$ are positive for $s>0$, and $u(0)=\nu(0)=0$. If there exists a continuous function $V:\mathbb{R}\times C\to\mathbb{R}$ such that:*

$$u(|\psi(0)|)\leq V(t,\psi)\leq\nu(|\psi|),$$
$$V'(t,\psi)\leq -w(|\psi(0)|),$$

then the solution $x=0$ of equation (1.15) is uniformly stable. If $w(s)>0$ for $s>0$, then the solution $x=0$ is uniformly asymptotically stable.

Numerical Examples and Tests

***Example* 1.13.** *We study the differential equation:*

$$\dot{x}(t)=ax(t),\quad x(0)=1 \tag{1.16}$$

Stability Analysis *Depending on the sign of a, we obtain:* ***Numerical Values of the Solution*** *We compute the values of $x(t)$ for selected values of t and three representative values of a:*

Table 1.1: Values of $x(t)$ for different a

t	$x(t)$ for $a = -1$	$x(t)$ for $a = 0$	$x(t)$ for $a = 0.7$
0	1.000	1.000	1.000
0.5	0.607	1.000	1.419
1	0.368	1.000	2.014
1.5	0.223	1.000	2.858
2	0.135	1.000	4.060
2.5	0.082	1.000	5.767
3	0.050	1.000	8.188

Graphical Representation

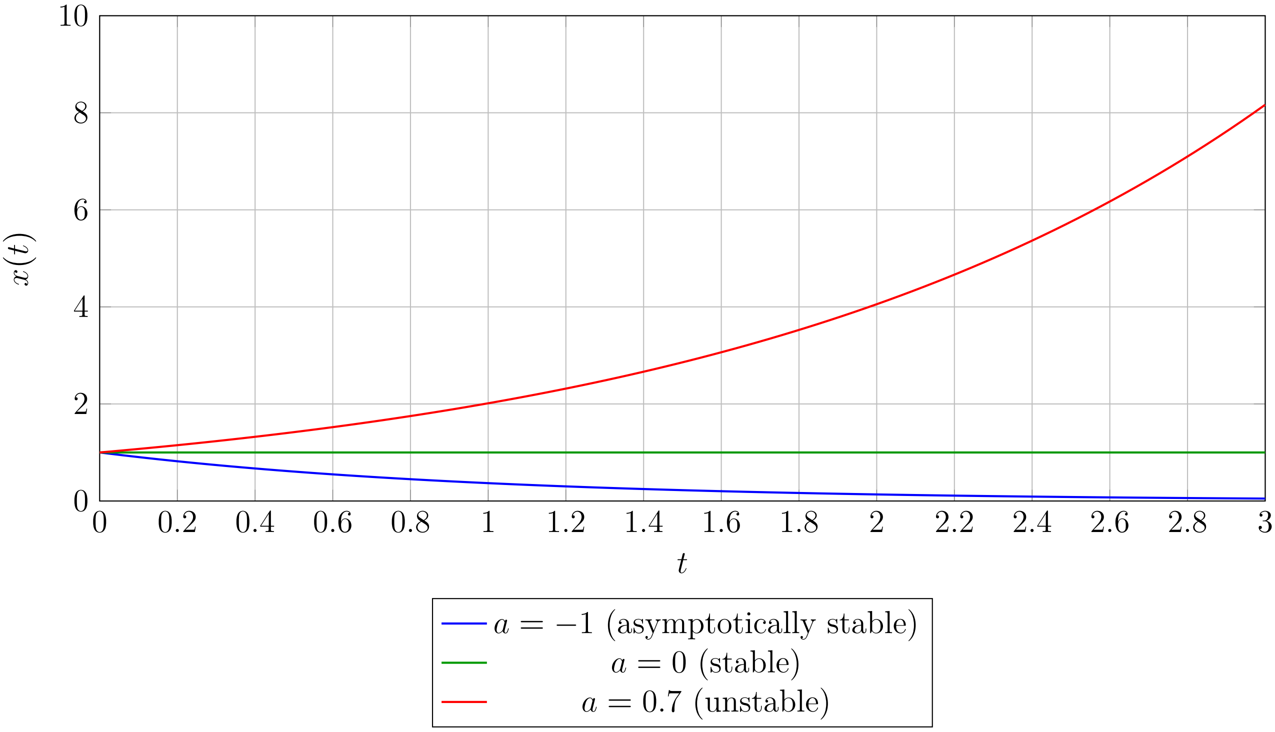


Figure 1.2: Solutions for different values of a.

Example 1.14. *Behavior of the Delay Differential Equation*
$\dot{x}(t) = ax(t-\tau)$

Case Analysis with Delay $\tau = 1$ *We analyze the delayed differential equation:*

$$\dot{x}(t) = ax(t-\tau), \quad t > 0, \quad x(s) = 1, -\tau \leq s \leq 0 \tag{1.17}$$

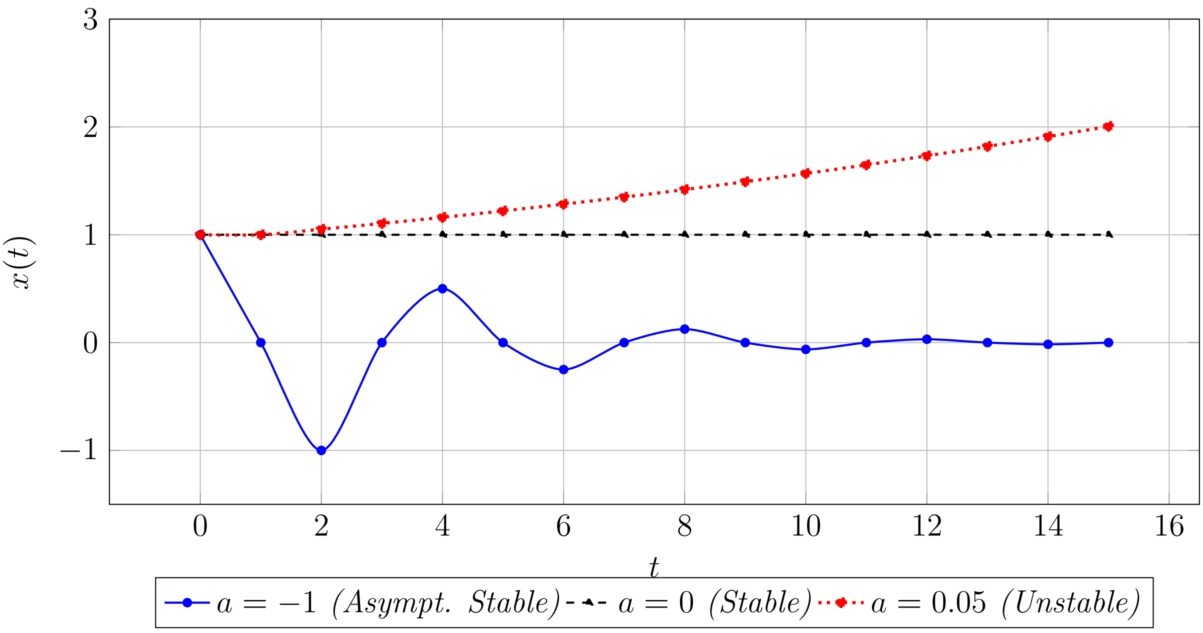
with different values of a *to understand the long-term behavior of the solutions.*

- ***Asymptotically Stable:*** $a = -1$ *results in oscillations that decay to zero.*
- ***Stable (not asymptotically):*** $a = 0$ *yields constant solutions.*
- ***Unstable:*** $a = 0.05$ *results in exponential-like growth.*

Table 1.2: Values of $x(t)$ for different a

t	$x(t)$ for $a=-1$	$x(t)$ for $a=0$	$x(t)$ for $a=0.05$
0	1.000	1.000	1.000
1	0.000	1.000	1.000
2	-1.000	1.000	1.051
3	0.000	1.000	1.105
4	0.500	1.000	1.161
5	0.000	1.000	1.221
6	-0.250	1.000	1.284
7	0.000	1.000	1.349
8	0.125	1.000	1.419
9	0.000	1.000	1.492
10	-0.063	1.000	1.568
11	0.000	1.000	1.648
12	0.031	1.000	1.732
13	0.000	1.000	1.819
14	-0.016	1.000	1.910
15	0.000	1.000	2.005

Plot of Solutions



1.5 Differential Systems

1.5.1 Generalities

Consider the linear system of ordinary differential equations of the form

$$\begin{aligned} x_1' &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + g_1(t), \\ x_2' &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + g_2(t), \\ &\vdots \\ x_n' &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + g_n(t), \end{aligned}$$

where the functions a_{ij}, g_i, $1 \le i \le n$, $1 \le j \le n$, *are continuous real-valued functions on an interval* $\mathcal{I}$. *Using vector and matrix notations, the above system is equivalent to the vector equation*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t), \tag{1.18}$$

where $\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\mathbf{x}' := \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$, *and* $A(t) := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $\mathbf{g}(t) := \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$

for t *in* $\mathcal{I}$. *Note that the matrix functions* A *and* $\mathbf{g}$ *are continuous on an interval* $\mathcal{I}$ *if and only* *if all their entries are continuous on $\mathcal{I}$.*

Definition 1.16. *We say the $n \times 1$ vector $\mathbf{y}$ is a solution of (1.18) on $\mathcal{I}$ if $\mathbf{y}$ is continuously differentiable on $\mathcal{I}$ and*

$$\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{g}(t)$$

for all t in $\mathcal{I}$.

We denote the space of n-tuple continuously differentiable functions $\mathbf{x} : \mathcal{I} \to \mathbb{R}^n$ by $C^1(\mathcal{I}, \mathbb{R}^n)$. Next, we define matrix norm.

Definition 1.17. *Let A be a matrix with entries a_{ij}, $1 \leq i \leq n$, $1 \leq j \leq n$, that is, $A = (a_{ij})$. We define the following possible norms on the matrix A.*

1. *The 1-norm:*
$$\|A\|_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$
(the maximum absolute column sum).

2. *The infinity norm:*
$$\|A\|_\infty = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$
(the maximum of the sums of the absolute values along each row).

3. *The Euclidean norm:*
$$\|A\|_E = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2}$$
(the square root of the sum of squares of all entries).

Theorem 1.11. ***[Gronwall's inequality [36]]*** *Let C be a nonnegative constant, and let u, v be nonnegative continuous functions on $[a, b]$ such that*

$$v(t) \leq C + \int_a^t v(s)u(s)\, ds, \quad a \leq t \leq b. \tag{1.19}$$

Then

$$v(t) \leq C e^{\int_a^t u(s)\, ds}, \quad a \leq t \leq b. \tag{1.20}$$

In particular, if $C = 0$, then $v = 0$.

Theorem 1.12. *Suppose $A(t)$ and $g(t)$ are continuous on some interval $a \leq t \leq b$. Then (1.18) has a unique solution $\phi(t)$ on the interval $a \leq t \leq b$ with $\phi(t_0) = \eta$ and $a \leq t_0 \leq b$.*

Proof. Applying Theorem 1.11 (Gronwall's inequality) □

Corollary 1.5.1. *If $A(t)$ and $g(t)$ are continuous on $\mathbb{R}$, then the unique solution $\phi(t)$ of (1.18) is defined for all $t \in \mathbb{R}$.*

In the next discussion, we try to estimate the error between two solutions. Let ϕ_1 and ϕ_2 be two solutions of (1.18) with $\phi_1(t_0) = \eta_1$ and $\phi_2(t_0) = \eta_2$, respectively. Then

$$\begin{aligned} |\phi_1(t) - \phi_2(t)| &\leq |\phi_1(t_0) - \phi_2(t_0)| + \left| \int_{t_0}^{t} A(s)[\phi_1(s) - \phi_2(s)]\, ds \right| \\ &\leq |\eta_1 - \eta_2| + \int_{t_0}^{t} \|A(s)\| |\phi_1(s) - \phi_2(s)|\, ds. \end{aligned}$$

Using Gronwall's inequality, we arrive at

$$|\phi_1(t) - \phi_2(t)| \leq |\eta_1 - \eta_2| e^{\int_{t_0}^{t} \|A(s)\|\, ds}, \tag{1.21}$$

which is an estimate of the error between the two solutions at two different initial conditions.

1.5.2 Fundamental matrix and exponential matrix

F*undamental matrix*

Solving the non-homogeneous equation (1.18) requires solving the homogeneous system

$$x'(t) = A(t)x(t), \tag{1.22}$$

where $A(t)$ is an $n \times n$ matrix of coefficients $a_{ij}(t)$, which are assumed to be continuous on an interval $\mathcal{I}$. Recall that a solution $x(t)$ of (1.22) is an n-tuple of C^1 functions $x_i : \mathcal{I} \to \mathbb{R}$. We adopt the notation

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

The solution x may be considered as a C^1 vector-valued function $x : \mathcal{I} \to \mathbb{R}^n$. The space of such functions is denoted by $C^1(I, \mathbb{R}^n)$. If $\mathcal{S}$ is the solution space of (1.22), then $\mathcal{S} \subset C^1(\mathcal{I}, \mathbb{R}^n)$.

Definition 1.18. ***(Fundamental set of solutions)*** *A set of n solutions of the linear differential system (1.22), all defined on the same open interval $\mathcal{I}$, is called a fundamental set of solutions on $\mathcal{I}$ if the solutions are linearly independent functions on $\mathcal{I}$.*

We have the following corollary.

Corollary 1.5.2. *Let $A(t)$ be an $n \times n$ matrix of continuous coefficients $a_{ij}(t)$ on an interval $\mathcal{I}$. If $\{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\}$ form a fundamental set of solutions on $\mathcal{I}$, then The general solution of (1.22) is given by*

$$x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + \cdots + c_n\varphi_n(t)$$

with constants c_i, $i = 1, 2, \ldots, n$.

Definition 1.19. ***[Fundamental matrix]*** *A matrix solution Φ is called a* fundamental matrix solution *(or, shortly,* fundamental matrix*) of (1.22) on an interval $\mathcal{I}$ if its columns form a fundamental set of solutions. If, in addition, $\Phi(t_0) = I$, then a fundamental matrix solution is called the* principal fundamental matrix solution *(or, shortly,* principal matrix*).*

Theorem 1.13. *([36]) The matrix Φ is a fundamental matrix of $x' = Ax$ at t_0 if and only if*

(a) Φ is a solution of $x' = Ax$, i.e. $\Phi^{'} = A\Phi$, and

(b) $\det \Phi(t_0) \neq 0$.

***Example* 1.15.** *We claim that the matrix*

$$\Phi(t) = \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{pmatrix}$$

is the fundamental matrix for the system

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

To see this, we have

$$\Phi'(t) = \begin{pmatrix} -e^{-t} & e^{-t} - te^{-t} \\ e^{-t} & -e^{-t} - (1-t)e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} & (1-t)e^{-t} \\ e^{-t} & -2e^{-t} + te^{-t} \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \Phi = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} & (1-t)e^{-t} \\ e^{-t} & te^{-t} - 2e^{-t} \end{pmatrix}.$$

Definition 1.20. *The* state transition matrix *for the homogeneous linear system (1.22) on the open interval I is the family of fundamental matrix solutions $t \mapsto \Psi(t, \tau)$ parametrized by $\tau \in \mathcal{I}$ such that $\Psi(\tau, \tau) = I$, where I denotes the $n \times n$ identity matrix.*

Proposition 1.1. *([18])[**the state transition matrix**] If $\Phi(t)$ is a fundamental matrix solution for the system (1.22) on I, then$\Psi(t, \tau) := \Phi(t)\Phi^{-1}(\tau)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman–Kolmogorov identities:*

$$\Psi(\tau, \tau) = I, \quad \Psi(t, s)\Psi(s, \tau) = \Psi(t, \tau)$$

and the identities:

$$\Psi(t, s)^{-1} = \Psi(s, t), \quad \frac{\partial}{\partial s}\Psi(t, s) = -\Psi(t, s)A(s).$$

Exponential matrix

The concept of the matrix exponential is considered one of the fundamental tools in applied mathematics, particularly in systems theory, control engineering, signal processing, and communication systems. Its role extends beyond merely solving linear time-invariant systems to encompass the complete characterization of system dynamics, frequency response analysis, and stability investigations.

Definition 1.21. *Let A be an $n \times n$ constant matrix. Then we define the* exponential matrix *function by $e^{A(t-t_0)}$, which is the solution of $x' = Ax$, and $e^{A(0)} = I$ (identity matrix).*

Thus we have already proved that

$$x(t) = e^{A(t-t_0)}x_0 \tag{1.23}$$

is the unique solution of

$$x'(t) = Ax(t), \quad x(t_0) = x_0, \tag{1.24}$$

for all $t \in \mathbb{R}$.

Definition 1.22. *For any square matrix $A \in \mathbb{C}^{n \times n}$ and scalar $t \in \mathbb{R}$, the matrix exponential is defined by the absolutely convergent series:*

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

where I denotes the $n \times n$ identity matrix.

Theorem 1.14 (Convergence). *The matrix exponential series converges absolutely for all $A \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}$.*

Proof. Using the matrix norm $\|\cdot\|$, we have:

$$\left\|\sum_{k=0}^{\infty}\frac{(At)^k}{k!}\right\| \leq \sum_{k=0}^{\infty}\frac{(\|A\||t|)^k}{k!} = e^{\|A\||t|} < \infty$$

□

Theorem 1.15. *[36] Let A and B be $n\times n$ constant matrices. Then:*

1. $\frac{d}{dt}e^{At} = Ae^{At}$, $t \in \mathbb{R}$;
2. $\det(e^{At}) \neq 0$, $t \in \mathbb{R}$, *and* e^{At} *is a fundamental matrix solution for (1.22);*
3. $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots$;
4. $e^{At}e^{As} = e^{A(t+s)}$, $t, s \in \mathbb{R}$;
5. $\left(e^{At}\right)^{-1} = e^{-At}$;
6. *If* $AB = BA$, *then* $e^{At}B = Be^{At}$;
7. *If* $AB = BA$, *then* $e^{At}e^{Bt} = e^{(A+B)t}$, $t \in \mathbb{R}$;
8. *If* P *is a nonsingular matrix, then* $e^{PBP^{-1}} = Pe^{B}P^{-1}$.

Methods for Computing e^{At}

Direct Series Expansion For simple matrices, the series can be computed explicitly by summing terms until convergence. However, this is often impractical for large matrices.

$$e^{At} = \sum_{k=0}^{\infty}\frac{(At)^k}{k!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \tag{1.25}$$

Diagonalizable Matrices If A is diagonalizable, i.e., $A = PDP^{-1}$, where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then:

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t}\end{pmatrix}P^{-1} \tag{1.26}$$

Jordan Canonical Form For non-diagonalizable matrices, A can be decomposed into its Jordan form $A = PJP^{-1}$, where J consists of Jordan blocks. Then:

$$e^{At} = Pe^{Jt}P^{-1} \tag{1.27}$$

where e^{Jt} is computed block-wise using:

$$e^{J_i t} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots \\ 0 & 1 & t & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.28}$$

Using Laplace Transforms The matrix exponential can also be computed using the inverse Laplace transform:

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \tag{1.29}$$

Putzer's Algorithm An alternative method for computing e^{At} without eigenvectors, using the Cayley-Hamilton theorem.

Theorem 1.16. *[36] Let $A(t)$ be an $n \times n$ matrix, and let $g(t)$ be an $n \times 1$ vector, both continuous on some interval I. Suppose $\mathbf{\Phi}$ is the fundamental matrix of the homogeneous system. Then $x(t)$ is a solution of*

$$x' = A(t)x + g(t), \quad x(t_0) = x_0, \ t \geq t_0, \tag{1.30}$$

on I if and only if x satisfies

$$x(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)x_0 + \int_{t_0}^{t} \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)g(s)ds, \tag{1.31}$$

where $t_0 \in I$ and $x_0 \in \mathbb{R}^n$.

Remark 1.3. *In the case where A is an $n \times n$ constant matrix, the exponential matrix is equivalent to*

$$e^{A(t-t_0)} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0),$$

which is the principal matrix. In this case, (1.31) reduces to or takes the form

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s)\, ds. \tag{1.32}$$

Example 1.16. *Solve the system:*

$$x'(t) = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} x + \begin{pmatrix} -15te^{-2t} \\ -4te^{-2t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

Using Putzer's algorithm, we find the matrix exponential:

$$e^{At} = \frac{1}{7} \begin{pmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{pmatrix}.$$

Note that $e^{A0} = I$*. Using formula (1.32), the solution is:*

$$\begin{aligned} x(t) &= \frac{1}{7} \begin{pmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} \\ &+ \int_0^t \frac{1}{7} \begin{pmatrix} e^{-2(t-s)} + 6e^{5(t-s)} & -2e^{-2(t-s)} + 2e^{5(t-s)} \\ -3e^{-2(t-s)} + 3e^{5(t-s)} & 6e^{-2(t-s)} + e^{5(t-s)} \end{pmatrix} \begin{pmatrix} -15se^{-2s} \\ -4se^{-2s} \end{pmatrix} ds \\ &= \frac{1}{14} \begin{pmatrix} (6 + 28t - 7t^2)e^{-2t} + 92e^{5t} \\ (-4 + 14t + 21t^2)e^{-2t} + 46e^{5t} \end{pmatrix}. \end{aligned}$$

Remark 1.4. *Another point of discussion is that if we integrate (1.22) from* t_0 *to* t*, then we get*

$$x(t) = e^{\int_{t_0}^t A(s)ds}.$$

Now let

$$\Phi(t) = e^{\int_{t_0}^t A(s)ds}. \tag{1.33}$$

For $\Phi(t)$ *to be a fundamental matrix solution, we must have*

$$A(t) \left(\int_{t_0}^t A(s)ds \right) = \left(\int_{t_0}^t A(s)ds \right) A(t). \tag{1.34}$$

Example 1.17. *Find the fundamental matrix* $\Phi(t)$ *for the system*

$$x' = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{where } A(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \text{ and } \Phi(t) = e^{\int_0^t A(s)ds}.$$

First, we compute:

$$A(t) \left(\int_0^t A(s)ds \right) = \left(\int_0^t A(s)ds \right) A(t) = \begin{pmatrix} t + t^3/2 & 3t^2/2 \\ 3t^2/2 & t + t^3/2 \end{pmatrix}.$$

The matrix $\int A(t)dt$ *has eigenvalues:*

$$\lambda_1 = t + \frac{t^2}{2} \quad \text{and} \quad \lambda_2 = t - \frac{t^2}{2}.$$

The corresponding eigenvectors (independent of t*) are:*

$$\lambda_1, K_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2, K_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let $H = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ *with inverse* $H^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. *We diagonalize* $\int_0^t A(s)ds$ *via:*

$$J = H^{-1}\left(\int_0^t A(s)ds\right)H = \begin{pmatrix} t + \frac{t^2}{2} & 0 \\ 0 & t - \frac{t^2}{2} \end{pmatrix}.$$

The exponential matrix of J *is:*

$$e^J = \begin{pmatrix} e^{t+\frac{t^2}{2}} & 0 \\ 0 & e^{t-\frac{t^2}{2}} \end{pmatrix}.$$

Finally, the fundamental matrix is:

$$\Phi(t) = e^{\int_0^t A(s)ds} = He^J H^{-1} = \frac{e^t}{2}\begin{pmatrix} e^{\frac{t^2}{2}} + e^{-\frac{t^2}{2}} & e^{\frac{t^2}{2}} - e^{-\frac{t^2}{2}} \\ e^{\frac{t^2}{2}} - e^{-\frac{t^2}{2}} & e^{\frac{t^2}{2}} + e^{-\frac{t^2}{2}} \end{pmatrix}.$$

1.5.3 Floquet Theory and stability systems

In 1883, French mathematician **Gaston Floquet** (1847–1920) made a pioneering contribution to differential equation theory through his development of **Floquet theory**, published in the *Annales Scientifiques de l'École Normale Supérieure*. This theory focuses on studying linear differential systems with periodic coefficients expressed by the formula:

$$x'(t) = A(t)x(t) \quad \text{where} \quad A(t+T) = A(t),$$

representing a fundamental mathematical framework for analyzing periodic systems. Floquet demonstrated that fundamental solutions to these systems take the form:

$$\Phi(t) = P(t)e^{Bt}$$

where $P(t)$ is a periodic function and B is a constant, while introducing the concept of **Floquet multipliers** used to assess system stability. The applications of this theory extend beyond theoretical frameworks to encompass broad practical domains including: Vibration physics in Mathieu oscillators, Electrical circuit engineering, Biological rhythm studies. Later expanded to include nonlinear systems through **Floquet-Lyapunov theory**, this work became a cornerstone of applied mathematics in the 20th century. Although Floquet's fame never reached that of contemporaries like Henri Poincaré, his theory proved particularly influential in fields such as: Quantum mechanics, Solid-state physics. In this section, we will begin the study of linear systems of the form

$$x'(t) = A(t)x, \quad x \in \mathbb{R}^n \tag{1.35}$$

where $t \to A(t)$ is a T-periodic continuous matrix-valued function. The main theorem in this section, Floquet's theorem, gives a canonical form for each fundamental matrix solution. This result will be used to show that there is a periodic time-dependent change of coordinates that transforms system (1.35) into a homogeneous linear system with constant coefficients. Floquet's theorem is a corollary of the following result about the range of the exponential map.

Theorem 1.17. *[18] If C is a nonsingular $n \times n$ matrix, then there is an $n \times n$ matrix B, possibly complex, such that $e^B = C$.*

Theorem 1.18. ***[Floquet's Theorem [18]]*** *If $\Phi(t)$ is a fundamental matrix solution of the T-periodic system (1.35), then, for all $t \in \mathbb{R}$,*

$$\Phi(t+T) = \Phi(t)\Phi^{-1}(0)\Phi(T).$$

In addition, for each possibly complex matrix B such that

$$e^{TB} = \Phi^{-1}(0)\Phi(T),$$

there is a possibly complex T-periodic matrix function $t \mapsto P(t)$ such that $\Phi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$.

Proof. Since the function $t \mapsto A(t)$ is periodic, it is defined for all $t \in \mathbb{R}$. Thus, by [[18], Theorem 2.4, p130] all solutions of the system are defined for $t \in \mathbb{R}$. If $\Psi(t) := \Phi(t+T)$, then Ψ is a matrix solution. Indeed, we have that

$$\dot{\Psi}'(t) = \dot{\Phi}'(t+T) = A(t+T)\Phi(t+T) = A(t)\Psi(t),$$

as required. Define

$$C := \Phi^{-1}(0)\Phi(T) = \Phi^{-1}(0)\Psi(0),$$

and note that C is nonsingular. The matrix function $t \mapsto \Phi(t)C$ is clearly a matrix solution of the linear system with initial value $\Phi(0)C = \Psi(0)$. By the uniqueness of solutions, $\Psi(t) = \Phi(t)C$ for all $t \in \mathbb{R}$. In particular, we have that

$$\Phi(t+T) = \Phi(t)C = \Phi(t)\Phi^{-1}(0)\Phi(T).$$

By Theorem 1.17, there is a matrix B, possibly complex, such that

$$e^{TB} = C.$$

If $P(t) := \Phi(t)e^{-tB}$, then

$$P(t+T) = \Phi(t+T)e^{-TB}e^{-tB} = \Phi(t)Ce^{-TB}e^{-tB} = \Phi(t)e^{-tB} = P(t),$$

Thus, we have $P(t+T) = P(t)$, and $\Phi(t) = P(t)e^{tB}$ as required. □

Stability Analysis of Linear Systems Consider the system of ordinary differential equations

$$x' = f(t, x), \tag{1.36}$$

where $f \in C([0,\infty) \times D, \mathbb{R}^n)$, and $D \subset \mathbb{R}^n$ is open. We say that a vector $x^* \in \mathbb{R}^n$ is an equilibrium, or constant solution, or equilibrium solution of (1.36) if

$$f(t, x^*) = 0, \quad 0 \le t < w,$$

for a positive constant w. In practice, it is easier to talk about a zero solution, or $x = 0$. To do so, we translate every nonzero equilibrium point to zero by the change of variables $\tilde{x}(t) = x(t) - x^*$, and from (1.36) we have that

$$\tilde{x}'(t) = (x(t) - x^*)' = x'(t) = f(t, \tilde{x}(t) + x^*).$$

In this case, $\tilde{x}(t) = 0$ is an equilibrium point of (1.36), which we may call the zero solution of (1.36). Thus, in most cases, we may require $f(t, 0) = 0$, $0 \le t < w$, when we talk about the zero solution. Next, we consider (1.36) with the initial condition $x(t_0) = x_0$ with the assumption that $f(t, 0) = 0$.

Definition 1.23. *The zero solution $x = 0$ of (1.36):*

(a) is stable (S) if for all $\varepsilon > 0$ and $t_0 \geq 0$, there is $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x(t_0)| < \delta$ implies $|x(t, t_0, x_0)| < \varepsilon$;

(b) is uniformly stable (US) if δ is independent of t_0;

(c) is unstable if it is not stable;

(d) is asymptotically stable (AS) if it is stable and $\lim_{t\to\infty} |x(t, t_0, x_0)| = 0$;

(e) is uniformly asymptotically stable (UAS) if it is US and there exists $\gamma > 0$ such that for each $\mu > 0$, there exists $T = T(\mu) > 0$ such that $|x(t_0)| < \gamma$, $t \geq t_0 + T$, implies $|x(t, t_0, x_0)| < \mu$.

Now we are in a position to discuss the stability of the zero solution of the linear system

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{1.37}$$

where $A(t)$ is an $n \times n$ matrix with continuous entries on $[0, \infty)$, using the concept of the fundamental matrix. We begin with the following theorem.

Theorem 1.19. *[36] Let $\Phi(t)$ be the fundamental matrix of (1.37). Then the zero solution of (1.37) is*

(a) stable if and only if there exists a positive constant M such that

$$|\Phi(t)| \leq M, \quad t \geq 0;$$

(b) asymptotically stable if and only if

$$|\Phi(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Theorem 1.20. *[36] Let $\Phi(t)$ be the fundamental matrix of (1.37). Then the zero solution of (1.37) is uniformly stable if and only if there exists a positive constant M such that*

$$|\Phi(t)\Phi^{-1}(s)| \leq M, \quad t \geq s \geq 0. \tag{1.38}$$

The next theorem provides necessary and sufficient conditions for the UAS.

Theorem 1.21. *[36] Let $\Phi(t)$ be the fundamental matrix of (1.37). Then the zero solution of (1.37) is uniformly asymptotically stable if and only if there exist positive constants M and β such that*

$$|\Phi(t)\Phi^{-1}(s)| \leq Me^{-\beta(t-s)}, \quad t \geq s \geq 0. \tag{1.39}$$

***Example* 1.18.** *The nonlinear system*

$$x'(t) = \begin{pmatrix} 0 & 1 \\ \frac{-2}{t^2} & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t > 0,$$

has the fundamental matrix

$$\Phi(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}, \quad t > 0.$$

By Theorem 1.19, its zero solution is unstable, since there is no constant M such that $\|\Phi(t)\| \leq M$.

***Example* 1.19.** *The system*

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t > 0,$$

has the fundamental matrix

$$\Phi(t) = e^{-t} \begin{pmatrix} \cos(t) & \sin(t) \\ -(\cos(t) + \sin(t)) & \cos(t) - \sin(t) \end{pmatrix}, \quad t > 0.$$

After some calculations, we can verify that for some positive constant K,

$$\|\Phi(t)\Phi^{-1}(s)\| \leq Ke^{-(t-s)}, \quad t \geq s \geq 0,$$

and by Theorem 1.21 the zero solution is UAS.

This section is concerned with homogeneous systems of the form

$$x'(t) = A(t)x(t), \tag{1.40}$$

where the matrix $A(t)$ is periodic with period T (constant), that is,

$$A(t+T) = A(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

For the rest of this section, we assume that $A(t)$ is a periodic matrix with period T. Our aim is to find an expression for the fundamental matrix in this case. Unlike the case of a constant coefficient matrix, the fundamental matrix cannot be expressed as

$$\Phi(t) = (\varphi_1, \ldots, \varphi_n),$$

where the φ_i, $i = 1, 2, \ldots, n$, satisfy

$$\varphi_i(t) = k_i e^{\lambda_i t},$$

and (λ_i, K_i), $i = 1, 2, \ldots, n$, are eigenpairs for the constant matrix A. To reinforce this notion, we offer the following example.

***Example* 1.20.** *Consider the nonlinear system*

$$x'(t) = \begin{pmatrix} \sin(t) & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then the corresponding $A(t)$ matrix is periodic with period $T = 2\pi$. Moreover, $A(t)$ has the eigenpairs

$$\left(\sin(t), \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Without thinking and not paying attention to the fact that A is not a constant matrix, using the obtained eigenpairs, we form the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{\sin(t)} & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

Definition 1.24. *Let $\Phi(t)$ be the fundamental matrix for the Floquet system (1.40). Then the eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ of $B := \Phi^{-1}(0)\Phi(T)$ are called the* Floquet multipliers *of the Floquet system (1.40).*

Theorem 1.22. *The zero solution of (1.40) is*

(i) stable if and only if the Floquet multipliers ρ satisfy $|\rho| \leq 1$ and there is a complete set of eigenvectors for any multiplier of modulus 1;

(ii) asymptotically stable if and only if $|\rho| < 1$ for every ρ.

Example 1.21. *Consider the periodic system with period π:*

$$x'(t) = \begin{pmatrix} -1 + \dfrac{1}{2}\sin^2(t) & -1 - \dfrac{1}{2}\cos(t)\sin(t) \\ 1 - \dfrac{1}{2}\cos(t)\sin(t) & -1 + \dfrac{1}{2}\cos^2(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By inspection, we found the fundamental matrix to be

$$\Phi(t) = \begin{pmatrix} -e^{-t/2}\sin(t) & e^{-t}\cos(t) \\ e^{-t/2}\cos(t) & e^{-t}\sin(t) \end{pmatrix}.$$

We encourage the reader to verify that

$$\Phi'(t) = A(t)\Phi(t).$$

Then

$$B := \Phi^{-1}(0)\Phi(T) = \Phi^{-1}(0)\Phi(\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -e^{-\pi} \\ -e^{-\pi/2} & 0 \end{pmatrix} = \begin{pmatrix} -e^{-\pi/2} & 0 \\ 0 & -e^{-\pi} \end{pmatrix},$$

and thus the eigenvalues of B are given by $\rho_1 = -e^{-\pi/2}$ and $\rho_2 = -e^{-\pi}$ with $|\rho_i| < 1$, $i = 1, 2$. By Theorem 1.22, the origin $(0, 0)$ is asymptotically stable.

Numerical Examples and Tests

Example 1.22. *(Neutral Delay Differential Systems)*
We consider NDDS of the form:

$$\frac{d}{dt}\left[x(t) + B\,x(t-\tau)\right] = A\,x(t) + C\,x(t-\tau),$$

where:

- $x(t) \in \mathbb{R}^2$ *is the state vector,*
- $B \in \mathbb{R}^{2\times 2}$ *is the neutral delay matrix,*
- $A, C \in \mathbb{R}^{2\times 2}$ *are constant system matrices,*
- $\tau = T = 1$ *is both the delay and the period.*

Two cases are considered: one leading to instability, and another exhibiting asymptotic stability.

- ***Case 1: Unstable System*** *The first system is given by:*

$$\frac{d}{dt}\left[x(t)+\begin{pmatrix}0.3 & 0\\ 0 & 0.3\end{pmatrix}x(t-1)\right]=\begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}x(t)+\begin{pmatrix}0 & 0\\ 0 & -0.5\end{pmatrix}x(t-1). \tag{1.41}$$

Table 1.3 presents the evolution of the state variables.

Table 1.3: State Evolution for Unstable System

Time (T)	$t=0$	$t=1T$	$t=2T$	$t=3T$	$t=4T$
$x_1(t)$	1.00	1.80	3.20	6.80	15.60
$x_2(t)$	0.00	1.20	3.80	9.00	22.50

Interpretation: *The continuous growth in both $x_1(t)$ and $x_2(t)$ indicates unbounded behavior. The system amplifies initial conditions exponentially, which is a clear manifestation of instability. No convergence towards equilibrium is observed.*

- ***Case 2: Asymptotically Stable System*** *The second system is defined by:*

$$\frac{d}{dt}\left[x(t)+\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix}x(t-1)\right]=\begin{pmatrix}0 & 1\\ -1 & -0.5\end{pmatrix}x(t). \tag{1.42}$$

The state evolution is summarized in Table 1.4.

Table 1.4: State Evolution for Asymptotically Stable System

Time (T)	$t=0$	$t=1T$	$t=2T$	$t=3T$	$t=4T$
$x_1(t)$	1.00	0.70	0.45	0.30	0.20
$x_2(t)$	0.00	0.40	0.25	0.15	0.10

Interpretation: *The trajectories of $x_1(t)$ and $x_2(t)$ display continuous decay over time. After four periods, both state variables have reduced to less than 20% of their initial values. This indicates* ***asymptotic stability****: the system returns to equilibrium as $t \to \infty$, regardless of small initial perturbations.*

- *Summary Comparison at $t = 4T$*

Table 1.5: Comparison of States at $t = 4T$

System Type	$x_1(4T)$	$x_2(4T)$
Unstable (Divergent)	15.60	22.50
Asymptotically Stable	0.20	0.10

- ***Graphical Representation***

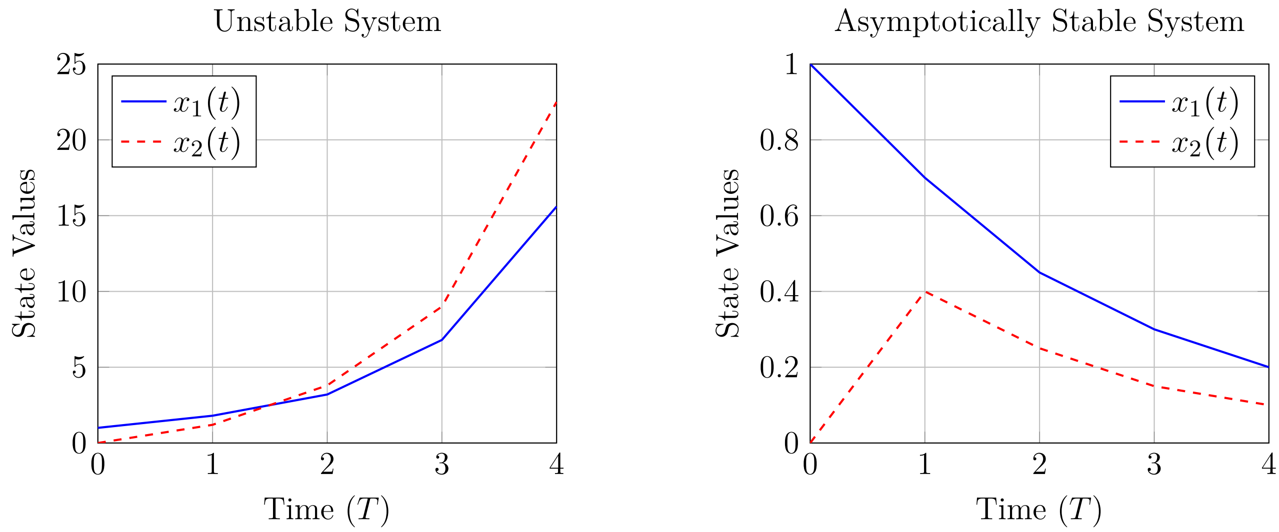


Figure 1.3: State evolution for both systems up to $t = 4T$.

Example 1.23. Analysis of a Series RLC Circuit Using Matrix Exponential *We study the second-order differential equation of a series RLC circuit:*

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \tag{1.43}$$

Given the parameters:

$$R = 2\,\Omega, \quad L = 1\,H, \quad C = 0.5\,F$$

The system can be written in state-space form:

$$\frac{d}{dt}\begin{pmatrix} q \\ i \end{pmatrix} = A\begin{pmatrix} q \\ i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \tag{1.44}$$

The eigenvalues of matrix A are:

$$\lambda_{1,2} = -1 \pm i$$

Thus, the solution using matrix exponential is:

$$e^{At} = e^{-t} \begin{pmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{pmatrix}$$

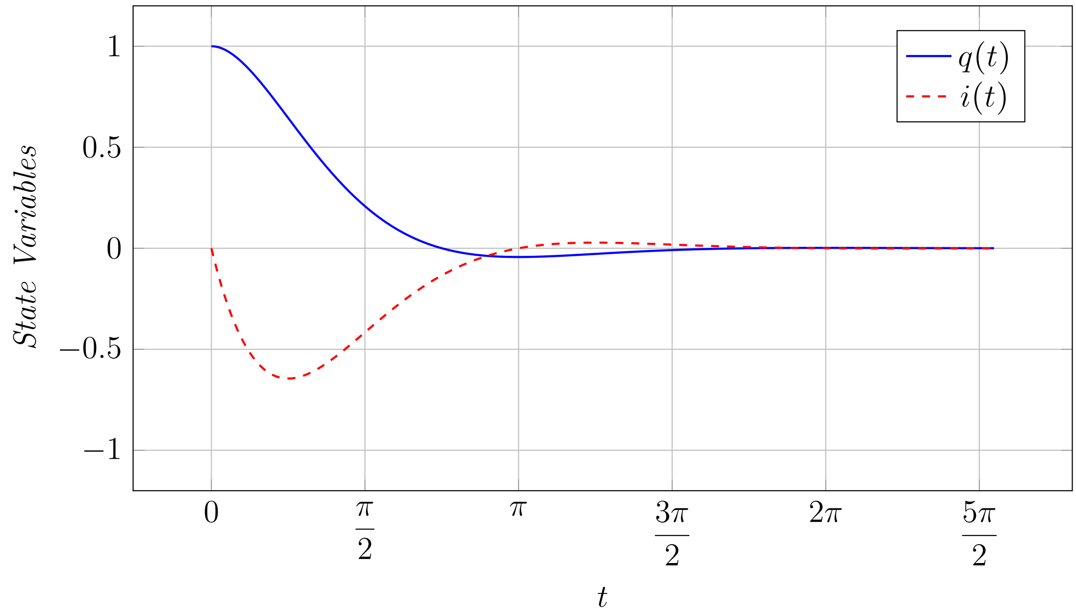
For the initial conditions $q(0) = 1\,C$, $i(0) = 0\,A$, the solution becomes:

$$q(t) = e^{-t}(\cos t + \sin t), \quad i(t) = -2e^{-t}\sin t$$

Numerical Results *The following table shows the computed values for selected time instants:*

t	$q(t)$	$i(t)$
0	1.0000	0.0000
$\frac{\pi}{2}$	0.2070	−0.8580
π	−0.0432	0.0000
2π	0.0018	0.0037

Graphical Representation



Example 1.24. ***Analysis of a Series RLC Circuit with Time Delay*** *We consider the following delay differential equation model:*

$$\frac{d}{dt}\begin{pmatrix} q(t) \\ i(t) \end{pmatrix} = A \begin{pmatrix} q(t) \\ i(t) \end{pmatrix} + B \begin{pmatrix} q(t-\tau) \\ i(t-\tau) \end{pmatrix} \tag{1.45}$$

where:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau = 1 \text{ second.}$$

The system includes a discrete time-delay affecting the current $i(t)$. ***Initial History Function*** *We assume constant history for $t \in [-1, 0]$:*

$$q(t) = 1, \quad i(t) = 0.$$

Numerical Solution *Since the system includes a time-delay term, no closed-form solution exists in terms of classical matrix exponential. The solution is approximated numerically (for example using DDE solvers like `dde23` in MATLAB).*

Computed Numerical Values

t	$q(t)$	$i(t)$
0	1.0000	0.0000
0.5	0.6975	−0.5827
1.0	0.2894	−0.8414
1.5	−0.0687	−0.6803
2.0	−0.2950	−0.2171
3.0	−0.1290	0.3321
4.0	0.2120	0.3157
5.0	0.3614	−0.0134

Stability Analysis *Without delay ($B = 0$), the eigenvalues of matrix A are:*

$$\lambda_{1,2} = -1 \pm i$$

which lie in the left-half complex plane. The system is therefore ***asymptotically stable*** *in the classical sense.*

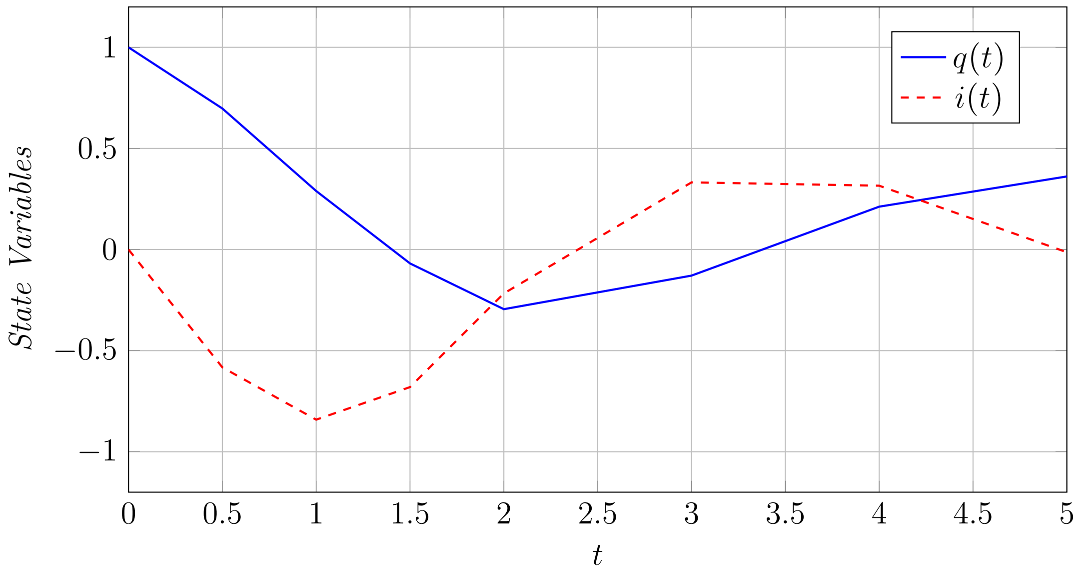
However, introducing time-delay may destabilize the system. The presence of the delayed feedback term makes the system stability depend on both delay size τ and feedback strength α.

For this case ($\tau = 1$, $\alpha = -1$): Numerical simulation shows damped oscillations without divergence. Therefore, the system is still ***delay-dependent asymptotically stable*** *for the given delay. A more rigorous analytical stability assessment would require characteristic quasipolynomial analysis:*

$$\det\left(\lambda I - A - Be^{-\lambda\tau}\right) = 0$$

which may be analyzed using the Lambert W function or Nyquist-type methods.

Graphical Representation



CHAPTER 2

Periodicity and stability in neutral non- linear differential equations

2.1 Introduction

Neutral-type differential equations represent a significant class of functional differential equations that incorporate delays not only in the state variables but also in their derivatives. This feature makes them particularly well-suited for modeling complex dynamical systems where memory effects and time delays critically influence the system's behavior. Such systems commonly arise in various engineering fields, including control theory, population dynamics, and notably, the analysis of electrical power transmission lines. Power transmission lines are inherently distributed parameter systems where signal propagation delays and dynamic feedback effects cannot be neglected. The presence of these delays and their influence on system stability call for mathematical models that go beyond classical ordinary differential equations. Neutral-type differential equations provide a rigorous framework to describe these phenomena accurately, capturing both the instantaneous and delayed effects on the system's evolution. In this chapter, we present important research published in the form of two articles. This study focuses on establishing sufficient conditions for the existence and stability of solutions to a class of neutral-type differential equations with time-varying delays. By utilizing powerful mathematical tools such as fixed-point theorems (particularly Krasnoselskii's theorem) and

Lyapunov-based methods, this study also aims to develop a solid theoretical foundation that ensures well-posedness and asymptotic stability of these systems. The theoretical insights gained here have practical implications for the design and stability analysis of power transmission systems, where ensuring reliable operation under varying load conditions and delay effects is critical. By better understanding the solution behavior of neutral-type models, engineers can devise control strategies that mitigate instability risks and improve overall system resilience.

In 2010, Ding and Li [19] studied a nonlinear neutral functional differential equation derived from a power transmission line model. By applying Krasnoselskii's fixed point theorem, they established sufficient conditions for the existence of asymptotically periodic solutions for this class of equations. This study was motivated by the challenges encountered in analyzing integrated electronic circuits (see 2.1), which consist of transmission lines with losses and nonlinear resistive loads exhibiting exponential voltage-current characteristics. In addition, the authors corrected one of the results presented in [V.G. Angelov [2], Lossy transmission lines terminated by R-loads with exponential V–I characteristics, Nonlinear Anal. RWA 8 (2007) 579–589]. As an application, they demonstrated the existence of periodic oscillations along the line when the load is a q–n diode. We particularly emphasize that this study primarily focuses on the simplified electrical transmission line circuit illustrated in Figure 2.1 (see 2.1), which consists of a periodic source $E(\varkappa)$ with resistance R_0, a capacitor C_0, and a nonlinear element characterized by the $i = \Gamma(u)$ relationship. Through analyzing this circuit, the research provides significant theoretical and practical contributions to understanding transmission line behavior. Building upon prior work by researchers such as Brayton (1966-1967) and Angelov (2007) see [2] ,

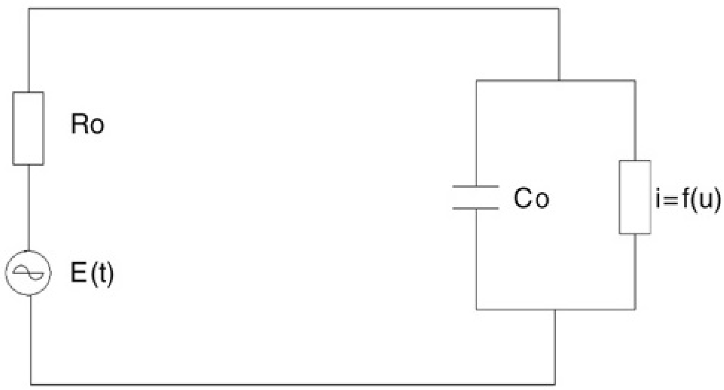


Figure 2.1: Fundamental circuit.

the study enhances stability conditions using the Lyapunov functional method while eliminating certain constraints present in previous studies. The application of Krasnoselskii's fixed point theorem to analyze the differential equations derived from the circuit model enabled the demonstration of stable periodic solutions in previously unaddressed cases.

Ding and Li [19] discussed the existence and stability of periodic solutions for the following neutral functional differential equation

$$\begin{aligned}\textbf{Problem (DL):}\quad & \frac{d}{d\varkappa}u(\varkappa) - r\frac{d}{d\varkappa}u(\varkappa-\varsigma) \\ & = r(\varkappa) - au(\varkappa) - aru(\varkappa-\varsigma) - c\Gamma(u(\varkappa)) + cr\Gamma(u(\varkappa-\varsigma)). \end{aligned} \tag{2.1}$$

In 2017, Mansouri et al. [31] studied the nonlinear neutral functional differential equation with variable delay

$$\begin{aligned}\textbf{Problem (MAD):}\quad & \frac{d}{d\varkappa}u(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\pounds(u(\varkappa-\varsigma(\varkappa))) \\ & = r(\varkappa) - a(\varkappa)u(\varkappa) - a(\varkappa)r(\varkappa)\pounds(u(\varkappa-\varsigma(\varkappa))) \\ & \quad - c(\varkappa)\Gamma(u(\varkappa)) + c(\varkappa)r(\varkappa)\Gamma(u(\varkappa-\varsigma(\varkappa))). \end{aligned} \tag{2.2}$$

These results extend previous work in [19]. The chapter is organized as: Section 2 proves existence of periodic solutions, Section 3 studies asymptotic stability, and Section 4 presents practical applications.

2.2 Existence of periodic solutions

In this section, we use the following notation:

- $C^1(\mathbb{R})$ denotes the space of all continuously differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$ denotes the space of all continuous functions $\phi : \mathbb{R} \to \mathbb{R}$
- $C_\omega = \{\phi \in C(\mathbb{R}) \mid \phi(\varkappa+\omega) = \phi(\varkappa)\}$ equipped with the supremum norm $\|\cdot\|_0$
- $C_\omega^1 = C^1(\mathbb{R}) \cap C_\omega$ equipped with the norm $\|\phi\|_1 = \|\phi\|_0 + \|\phi'\|_0$ on a periodic interval.

2.2.1 Problem(DL)

We will use the following fundamental results:

Lemma 2.1. *If $a \neq 0$ and $\Gamma \in C_\omega$, then the scalar differential equation*

$$x'(\varkappa) = ax(\varkappa) + \Gamma(\varkappa)$$

has a unique ω-periodic solution given by

$$x(\varkappa) = (1 - e^{a\omega})^{-1} \int_{\varkappa}^{\varkappa+\omega} e^{a(\varkappa+\omega-s)}\Gamma(s)ds.$$

Proof. The proof is standard and can be found in many ordinary differential equations textbooks. □

Theorem 2.1. *[19] Suppose $\Gamma \in C^1(\mathbb{R})$ and $q \in C^1_T$. If there exists a constant $H > 0$ such that*

$$\frac{\sup_{|u|\leq H} |\Gamma(u)|}{H} < \frac{a}{c}, \tag{2.3}$$

and

$$|r| < \frac{1 - \frac{c}{a}\frac{\sup_{|u|\leq H}|\Gamma(u)|}{H}}{3 + \frac{c}{a}\frac{\sup_{|u|\leq H}|\Gamma(u)|}{H}} \quad \text{and} \quad \|q\|_0 < (1 - 3|r|)aH - c(1 + |r|) \sup_{|u|\leq H} |\Gamma(u)|, \tag{2.4}$$

then equation (2.1) has a T-periodic solution.

Proof. From conditions (2.3) and (2.4), we can find $L > 0$ sufficiently small to satisfy

$$\left(\frac{1}{a} + 2L\right)\|q\|_0 + 3|r|H + 4aL|r|H + \left(\frac{c}{a} + 2L\right)(1 + |r|) \sup_{|u|\leq H} |\Gamma(u)| \leq H. \tag{2.5}$$

Define the change of variables:

$$v(\varkappa) = u(L\varkappa), \quad \mu = \frac{\varsigma}{L}, \quad q(L\varkappa) = q_1(\varkappa), \quad \omega = \frac{T}{L}.$$

Then Eq. (2.1) transforms to:

$$v'(\varkappa) - qv'(\varkappa - \mu) = Lp_1(\varkappa) - aLv(\varkappa) - aLqv(\varkappa - \mu) - bLf(v(\varkappa)) + bLqf(v(\varkappa - \mu)), \tag{2.6}$$

where $q_1(\varkappa) \in C^1_\omega$ with $\|q\|_0 = \|q_1\|_0$. Let:

$$\mathbb{S} = \{\phi \in C^1(\mathbb{R}) \mid \|\phi\|_1 = \|\phi\|_0 + \|\phi'\|_0 < \infty\},$$
$$\mathbb{M} = \{\phi \in C^1_\omega \mid \|\phi\|_1 \leq H\}.$$

Then $\mathbb{M}$ is a bounded closed convex subset of the Banach space $\mathbb{S}$. For $\phi \in \mathbb{M}$, consider the nonhomogeneous equation:

$$\begin{aligned} v'(\varkappa) &= -aLv(\varkappa) + Lq_1(\varkappa) - aLr\phi(\varkappa - \mu) \\ &\quad - cL\Gamma(\phi(\varkappa)) + cLr\Gamma(\phi(\varkappa - \mu)) + r\phi'(\varkappa - \mu). \end{aligned} \tag{2.7}$$

By Lemma (2.1), this has a unique ω-periodic solution:

$$\begin{aligned} v(\varkappa) = (1 - e^{-aL\omega})^{-1} \int_{\varkappa}^{\varkappa+\omega} e^{-aL(\varkappa+\omega-s)} \cdot \Big[& Lq_1(s) - aLr\phi(s-\mu) - cL\Gamma(\phi(s)) \\ & + cLr\Gamma(\phi(s-\mu)) + r\phi'(s-\mu) \Big] ds. \end{aligned}$$

Define operators $\mathcal{A}$ and $\mathcal{B}$ by:

$$\begin{aligned} (\mathcal{A}\phi)(\varkappa) &= (1 - e^{-aL\omega})^{-1} \int_{\varkappa}^{\varkappa+\omega} e^{-aL(\varkappa+\omega-s)} \Big[Lq_1(s) - 2aLr\phi(s-\mu) \\ &\qquad - cL\Gamma(\phi(s)) + cLr\Gamma(\phi(s-\mu)) \Big] ds, \\ (\mathcal{B}\phi)(\varkappa) &= r\phi(\varkappa - \mu). \end{aligned}$$

Verifying that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Theorem (1.5) shows there exists $\phi \in \mathbb{M}$ such that $\phi = \mathcal{A}\phi + \mathcal{B}\phi$, which is an ω-periodic solution of (2.7). Since $u(L\varkappa) = v(\varkappa)$ and $q(L\varkappa) = q_1(\varkappa)$, Eq. (2.1) has a T-periodic solution. □

Remark 2.1. *If $q(\varkappa)$ is a nonconstant periodic function, then Eq. (2.1) has a nonconstant periodic solution. Our approach provides a method to detect periodic oscillations along the transmission line that differs from Lopes' technique.*

Remark 2.2. *In [2], Angelov attempted to prove the existence of a T-periodic solution for equation. (2.1) using the contraction mapping principle. However, the constructed map was not a self-mapping. We resolve this issue by employing Krasnoselskii's fixed point theorem instead.*

Remark 2.3. *For equation (2.1), where $a = \dfrac{1}{Z_0 C_0}$ and $c = \dfrac{1}{C_0}$, we have $\dfrac{a}{c} = \dfrac{1}{Z_0}$. Thus, the key condition in Theorem 2.1 requires the existence of $H > 0$ such that*

$$\sup_{|u| \le H} \frac{|\Gamma(u)|}{H} < \frac{1}{Z_0}.$$

This inequality plays a crucial role in establishing the existence of periodic solutions.

2.2.2 Problem(MAD)

Since we are investigating the existence of periodic solutions for equation (2.2), it is natural to impose the following periodicity conditions. The system parameters satisfy:

$$r(\varkappa+T)=r(\varkappa),\quad q(\varkappa+T)=q(\varkappa),\quad a(\varkappa+T)=a(\varkappa),\quad c(\varkappa+T)=c(\varkappa),$$
$$\varsigma(\varkappa+T)=\varsigma(\varkappa).$$

The nonlinear function $\pounds:\mathbb{R}\to\mathbb{R}$ satisfies:

1. **Global Lipschitz condition**: There exists $k>0$ such that

$$|\pounds(x)-\pounds(y)|\leq k\|x-y\|_0 \quad \forall x,y\in\mathbb{R} \tag{2.8}$$

2. **Continuous differentiability**: $\pounds\in C^1(\mathbb{R})$ with $\|\pounds'\|_0=l_1$

Lemma 2.2. ***[Periodic Solution of Linear Equation]*** *If $a(\varkappa)\neq 0$ for all $\varkappa$and$\Gamma\in C_\omega$, then the scalar differential equation*

$$x'(\varkappa)+a(\varkappa)x(\varkappa)=\Gamma(\varkappa) \tag{2.9}$$

has a unique ω-periodic solution given by

$$x(\varkappa)=\left(1-e^{-\int_\varkappa^{\varkappa+\omega}a(u)du}\right)^{-1}\int_\varkappa^{\varkappa+\omega}e^{-\int_s^{\varkappa+\omega}a(u)du}\Gamma(s)ds. \tag{2.10}$$

Proof. This result can be found in standard ODE references. □

Using Lemmas (2.2) and Theorem (1.5), we establish the existence of periodic solutions for equation (2.2). Through an appropriate time-scaling transformation with parameter $l>0$, we consider the equivalent equation:

$$\begin{aligned}\frac{d}{dt}v(\varkappa)-lr_1(\varkappa)\frac{d}{dt}\pounds(v(\varkappa-\tau(\varkappa)))\\ &=lq_1(\varkappa)-la_1(\varkappa)v(\varkappa)-la_1(\varkappa)r_1(\varkappa)\pounds(v(\varkappa-\tau(\varkappa)))\\ &-lc_1(\varkappa)\Gamma(v(\varkappa))+lc_1(\varkappa)r_1(\varkappa)\Gamma(v(\varkappa-\tau(\varkappa)))\end{aligned} \tag{2.11}$$

with the scaling transformations:

$$v(\varkappa)=u(l\varkappa),\quad \tau(\varkappa)=\frac{\varsigma(l\varkappa)}{l},\quad r_1(\varkappa)=r(l\varkappa),$$
$$q_1(\varkappa)=q(l\varkappa),a_1(\varkappa)=a(l\varkappa),c_1(\varkappa)=c(l\varkappa),\omega=\frac{T}{l}$$

Theorem 2.2. ***[Existence of Periodic Solutions** [31]] Assume the following conditions hold: $\Gamma \in C^1(\mathbb{R})$ with Lipschitz constant k for $\pounds$*

$r_1, q_1, a_1, c_1 \in C^1_\omega$ with $\|r_1'\| = \beta$

The delay satisfies $\|1 - \tau'(\varkappa)\| = l_2$

If there exist constants $\rho \in (0,1)$ and $H > 0$ such that:

$$lk\left(\|r_1\|_0(1 + l_2 H) + \beta\right) \leq \rho \tag{2.12}$$

$$\sup_{|u|\leq H} |\Gamma(u)| < \frac{\theta_1 H - \theta_3}{k\|c_1\|_0} \tag{2.13}$$

$$\|r_1\|_0 < \frac{\theta_1 - \left(3 + \frac{\|c_1\|_0 \sup_{|u|\leq H}|\Gamma(u)|}{H}\right)}{\theta_2 + \frac{\|c_1\|_0 \sup_{|u|\leq H}|\Gamma(u)|}{H}} \tag{2.14}$$

and the norm condition:

$$\|q_1\|_0 < (\theta_1 - \theta_2\|r_1\|_0)H - \|c_1\|_0(1 + \|r_1\|_0)\sup_{|u|\leq H}|\Gamma(u)| - \theta_3 \tag{2.15}$$

where the parameters are defined as:

$$\begin{aligned}
\theta_1 &= \frac{\frac{1}{l} - k\beta}{1 + \alpha\omega M(1 + l\|a_1\|_0)} \\
\theta_2 &= \frac{k + l_1 l_2}{1 + \alpha\omega M(1 + l\|a_1\|_0)} + 2k\|a_2\|_0 \\
\theta_3 &= \frac{(\|r_1\|_0 + \beta)|\pounds(0)|}{1 + \alpha\omega M(1 + l\|a_1\|_0)} + 2\|a_2\|_0\|r_1\|_0|\pounds(0)| \\
2a_2(s)r_1(s) &= (l+1)a_1(s)r_1(s) + r_1'(s) \\
M &= \sup_{\varkappa\in\mathbb{R}}\left|\int_{\varkappa}^{\varkappa+\omega} e^{\int_s^{\varkappa+\omega} la_1(u)du} ds\right| \\
\alpha &= \left(1 - e^{-\int_{\varkappa}^{\varkappa+\omega} la_1(u)du}\right)^{-1}
\end{aligned}$$

Then equation (2.2) admits a T-periodic solution.

2.3 Asymptotic stability of the equilibrium and periodic solutions

2.3.1 Problem(DL)

When the source $E(\varkappa)$ is constant and the equation $q - a(1+r)u = c(1-r)\Gamma(u)$ has a unique solution u^*, then u^* is the equilibrium of Eq. (2.1). In this case, Eq. (2.1) can be transformed into:

$$u'(\varkappa) - ru'(\varkappa - \varsigma) = -au(\varkappa) - aru(\varkappa - \varsigma) - c\pounds(u(\varkappa)) + cr\pounds(u(\varkappa - \varsigma)), \tag{2.16}$$

where quad $\pounds(u) = \Gamma(u + u^*) - \Gamma(u^*)$ satisfies $\pounds(0) = 0$.
We now examine the stability of the zero solution for Eq. (2.16). First, we recall the following lemma from the literature:

Lemma 2.3 (Theorem 5.2 in [21], p. 281) *Suppose:*

- *D is stable,*
- *$D, L : C \to \mathbb{R}^n$ are linear and continuous operators,*
- *The zero solution of the neutral functional differential equation (NFDE)(D, L) is uniformly asymptotically stable.*

If $F, G : C \to \mathbb{R}^n$ are continuous with their first derivatives F_ϕ, G_ϕ, and satisfy:

- *$F(0) = G(0) = 0$,*
- *$F_\phi(0) = G_\phi(0) = 0$,*
- *$G(\phi)$ is independent of $\phi(0)$,*

then the zero solution of the equation

$$\frac{d}{dt}[Dx_\varkappa - G(x_\varkappa)] = Lx_\varkappa + F(x_\varkappa), \quad \varkappa \geq \sigma \tag{2.17}$$

is exponentially asymptotically stable.

Theorem 2.3. *[19] Suppose $\pounds \in C^1(\mathbb{R})$ satisfies a locally Lipschitz condition with $\pounds(0) = 0$. Then the zero solution of Eq. (2.16) is exponentially asymptotically stable.*

Proof. For any $\phi \in C = C([-\varsigma, 0]; \mathbb{R})$, define the operators:

$$D\phi = \phi(0) - r\phi(-\varsigma),$$
$$L\phi = -a\phi(0) - aq\phi(-\varsigma),$$
$$F\phi = -c\pounds(\phi(0)) + cr\pounds(\phi(-\varsigma)).$$

Here D is stable, and both D and L are linear continuous operators. Consider the equation $\frac{d}{dt}Du_{\varkappa} = Lu_{\varkappa}$ and define the Lyapunov functional:

$$V(\phi) = (D\phi)^2 + 2ar^2 \int_{-r}^{0} \phi^2(\theta)d\theta.$$

The derivative of V along solutions satisfies:

$$\begin{aligned} \dot{V}(\phi) &= 2(D\phi)(-a\phi(0) - aq\phi(-\varsigma)) + 2ar^2(\phi^2(0) - \phi^2(-\varsigma)) \\ &= -2a(1 - \varsigma^2)\phi^2(0) \le 0. \end{aligned}$$

By Theorem 8.1 in [[21], P. 293], the zero solution of

$$u'(\varkappa) - qu'(\varkappa - \varsigma) = -au(\varkappa) - aqu(\varkappa - \varsigma)$$

is uniformly asymptotically stable.
Moreover, the Fréchet derivative of $\Gamma is : F_{\phi}u = -cg'(\phi(0))u(0) + crg'(\phi(-\varsigma))u(-\varsigma)$, so $F(0) = F_{\phi}(0) = 0$. Since £is locally Lipschitz and $\pounds(0) = 0$, all conditions of Lemma (2.3) are satisfied. Therefore, the zero solution of Eq. (2.16) is exponentially asymptotically stable. □

Assume that the source depicted in Fig(2.1) is a T-periodic function $E(\varkappa)$, and that the assumptions of Theorem (2.1) are satisfied. Under these conditions, Eq.(2.16) admits a T-periodic solution denoted by $u^*(\varkappa)$. Introducing the change of variables $v(\varkappa) = u(\varkappa) - u^*(\varkappa)$, Eq.(2.16) can be rewritten in the form:

$$\begin{aligned} v'(\varkappa) - rv'(\varkappa - \varsigma) = -av(\varkappa) - arv(\varkappa - \varsigma) - cT\left(\Gamma\left(v(\varkappa) + u^*(\varkappa)\right) - \Gamma\left(u^*(\varkappa)\right)\right) \\ + cr\left(\Gamma\left(v(\varkappa - \varsigma) + u^*(\varkappa - \varsigma)\right) - \Gamma\left(u^*(\varkappa - \varsigma)\right)\right). \end{aligned} \tag{2.18}$$

It is evident that $v(\varkappa) = 0$ is a solution of Eq.(2.18). Our objective now is to prove that this zero solution is asymptotically stable. To this end, we apply Krasnoselskii's fixed point theorem. We define the space S as the Banach space of bounded continuous functions $\varphi : [-r, \infty) \to \mathbb{R}$, equipped with the supremum norm $\|\cdot\|$. Furthermore, for a given initial function φ, we define its norm by

$$\|\varphi\| = \sup_{\varkappa \in [-r,0]} |\varphi(\varkappa)|,$$

which should not lead to confusion with the supremum norm defined on the entire space S.

Theorem 2.4. *[19] Assume all conditions of Theorem (2.1) are satisfied and Γ is locally Lipschitz. If there exists $Q > H$ such that:*

$$\frac{c}{a} \sup_{|x| \leq H+Q} |\Gamma(x)| < Q - H, \tag{2.19}$$

and the following inequalities hold:

$$|r| < \frac{Q - H - \frac{c}{a} \sup\limits_{|x| \leq H+Q} |\Gamma(x)|}{3Q + H + \frac{c}{a} \sup\limits_{|x| \leq H+Q} |\Gamma(x)|}, \tag{2.20}$$

$$\|\psi\| \leq \frac{Q - 3|r|Q - (1 + |r|)H - \frac{c}{a} \sup\limits_{|x| \leq H+Q} |\Gamma(x)|}{1 + |r|}, \tag{2.21}$$

then the solution $v(\varkappa)$ of Eq. (2.16) satisfies $v(\varkappa) \to 0$ as $\varkappa \to \infty$.

Proof Continued. From conditions (2.19) and (2.20), we derive the key inequality:

$$3|r|Q + (1 + |r|)\|\psi\| + (1 + |r|)H + \frac{c}{a}(1 + |r|) \sup_{|x| \leq H+Q} |\Gamma(x)| \leq Q. \tag{2.22}$$

Given the initial function ψ, there exists a unique solution $v_\psi(\varkappa)$ to Eq. (2.18). Define the set:

$$M_\psi = \{\phi \in S \mid \|\phi\| \leq Q,\, \phi_0 = \psi,\, |\phi(\varkappa)| \to 0 \text{ as } \varkappa \to \infty\},$$

which is a bounded, convex, closed subset of S. Rewriting Eq. (2.18) in integrated form:

$$\begin{aligned} v(\varkappa) &= [\psi(0) - r\psi(\varkappa - \varsigma)]e^{-a\varkappa} + qv(\varkappa - \varsigma) \\ &\quad + \int_0^{\varkappa} \Big[-2aqv(s - \varsigma) - c\Big(\Gamma(v(s) + u^*(s)) - \Gamma(u^*(s))\Big) \\ &\quad + bq\Big(\Gamma(v(s - \varsigma) + u^*(s - \varsigma)) - \Gamma(u^*(s - \varsigma))\Big)\Big] e^{-a(\varkappa - s)} ds. \end{aligned} \tag{2.23}$$

Define operators A and c on M_ψ by:

$$(A\phi)(\varkappa) = \begin{cases} 0, & \varkappa \in [-\varsigma, 0] \\ \displaystyle\int_0^{\varkappa} \Big[-2aq\phi(s - \varsigma) - c\Big(\Gamma(\phi(s) + u^*(s)) - \Gamma(u^*(s))\Big) \\ \quad +cr\Big(\Gamma(\phi(s - \varsigma) + u^*(s - r)) - \Gamma(u^*(s - \varsigma))\Big)\Big] e^{-a(\varkappa - s)} ds, & \varkappa \geq 0 \end{cases}$$

$$(B\phi)(\varkappa) = \begin{cases} \psi(\varkappa), & \varkappa \in [-\varsigma, 0] \\ [\psi(0) - r\psi(-\varsigma)]e^{-a\varkappa} + r\phi(\varkappa - \varsigma), & \varkappa \geq 0 \end{cases}$$

Applying Krasnoselskii's fixed point theorem. Thus, there exists $\phi \in M_\psi$ such that $(A + B)\phi = \phi$, which is a solution of (2.18). By uniqueness, $v_\psi(\varkappa) = \phi(\varkappa) \to 0$ as $\varkappa \to \infty$. □

Theorem 2.5. *Assume Γ satisfies the locally Lipschitz condition, and the constants H from Theorem 2.1 and Q from Theorem 2.4 exist. If there exists $P > 0$ such that*

$$|\Gamma(v(\varkappa) + u^*(\varkappa)) - \Gamma(u^*(\varkappa))| < P|v(\varkappa)|, \tag{2.24}$$

and the stability condition

$$1 - 3|r| - \frac{c}{a}(1 + |r|)P > 0 \tag{2.25}$$

holds, then the zero solution of Eq. (2.18) is stable.

Proof. Given the solution ϕ satisfying the integral equation:

$$\phi(\varkappa) = [\psi(0) - r\psi(-\varsigma)]e^{-at} + r\phi(\varkappa - \varsigma) + \int_0^{\varkappa} G(s, \phi)e^{-a(\varkappa - s)}ds,$$

where

$$\begin{aligned} G(s, \phi) = &-2aq\phi(s - \varsigma) - c\Big[\Gamma(\phi(s) + u^*(s)) - \Gamma(u^*(s))\Big] \\ &+ cq\Big[\Gamma(\phi(s - \varsigma) + u^*(s - \varsigma)) - \Gamma(u^*(s - \varsigma))\Big] \end{aligned}$$

we estimate the norm:

$$\|\phi\| \leq (1 + |r|)\|\psi\| + |r|\|\phi\| + \frac{1}{a}\left[2a|r|\|\phi\| + c(1 + |r|)P\|\phi\|\right]$$

$$\left[1 - 3|r| - \frac{c}{a}(1 + |r|)P\right]\|\phi\| \leq (1 + |r|)\|\psi\|$$

For any $\epsilon > 0$, choose $\delta = \dfrac{\epsilon[1 - 3|r| - \frac{c}{a}(1 + |r|)P]}{1 + |r|}$. Then $\|\psi\| < \delta$ implies:

$$\|\phi\| \leq \frac{1 + |r|}{1 - 3|r| - \frac{c}{a}(1 + |r|)P}\delta = \epsilon,$$

proving stability of the zero solution. □

2.3.2 Problem(MAD)

Let $u^*(\varkappa)$ be a periodic equilibrium solution of (2.2). Consider the perturbation $v(\varkappa) = u(\varkappa) - u^*(\varkappa)$, which transforms (2.2) into:

$$\begin{aligned} &\frac{d}{dt}v(\varkappa) - r(\varkappa)\frac{d}{dt}\Big(\pounds(v(\varkappa - \varsigma(\varkappa)) + u^*(\varkappa - \varsigma(\varkappa))) - \pounds(u^*(\varkappa - \varsigma(\varkappa)))\Big) \\ &\qquad = -a(\varkappa)v(\varkappa) - a(\varkappa)r(\varkappa)\Big(\pounds(v(\varkappa - \varsigma(\varkappa)) + u^*(\varkappa - \varsigma(\varkappa))) - \pounds(u^*(\varkappa - \varsigma(\varkappa)))\Big) \\ &\qquad - c(\varkappa)\Big(\Gamma(v(\varkappa) + u^*(\varkappa)) - \Gamma(u^*(\varkappa))\Big) \\ &\qquad + c(\varkappa)r(\varkappa)\Big(\Gamma(v(\varkappa - \varsigma(\varkappa)) + u^*(\varkappa - \varsigma(\varkappa))) - \Gamma(u^*(\varkappa - \varsigma(\varkappa)))\Big) \end{aligned} \tag{2.26}$$

The zero solution $v \equiv 0$ corresponds to the periodic solution u^* of the original system. We establish asymptotic stability using the following setup:

- **Function Space**: Let S be the Banach space of bounded continuous functions $\phi : [m(0), \infty) \to \mathbb{R}$ equipped with the supremum norm $\|\cdot\|$, where $m(0) = \inf\{\varkappa - r(\varkappa) \mid \varkappa \geq 0\}$.
- **Initial Conditions**: For initial data $\psi \in C([m(0), 0], \mathbb{R})$, we define $\|\psi\| = \sup_{\varkappa \in [m(0),0]} |\psi(\varkappa)|$.

Theorem 2.6. *[31] Assume all conditions of Theorem (2.2) hold*

Γ satisfies a local Lipschitz condition with constant k

The integral condition: $\int_0^{\varkappa} a(u)du > 0$ and $e^{-\int_0^{\varkappa} a(u)du} \to 0$ as $\varkappa \to \infty$

The delay condition: $\varkappa - r(\varkappa) \to \infty$ as $\varkappa \to \infty$

The norm constraint: $k\|r\| < 1$

If there exists $R > H$ such that:

$$\|c\| \sup_{|u| \leq H+R} |\Gamma(u)| < \left(\frac{1}{\delta} - k\beta\right) R - (2k\beta + \theta_1)H - \theta_3 \tag{2.27}$$

$$\|r\| < \frac{(1 - k\delta\beta)R - 2k\delta\beta H - \delta\|c\| \sup_{|u|\leq H+R} |\Gamma(u)| - \delta(\theta_1 H + \theta_3)}{(2\delta\|a\| + 1)kR + 4k\delta\|a\|H + \delta\|c\| \sup_{|u|\leq H+R} |\Gamma(u)| + \delta(\theta_1 H + \theta_3)} \tag{2.28}$$

and the initial data satisfies:

$$\|\psi\| \leq \frac{1}{1 + k\|r\|} \begin{bmatrix} R\Big(1 - k\delta(2\|a\|\|r\| + \beta) - k\|r\|\Big) \\ - 2k\delta(2\|a\|\|r\| + \beta)H \\ - \delta\|c\|(1 + \|r\|) \sup_{|u|\leq H+R} |\Gamma(u)| \\ - \delta(1 + \|r\|)(\theta_1 H + \theta_3) \end{bmatrix} \tag{2.29}$$

where δ satisfies:

$$\sup_{\varkappa \geq 0} \left| \int_0^{\varkappa} e^{\int_{-s}^{\varkappa} a(u)du} ds \right| \leq \delta$$

then the solution $v(\varkappa)$ of (2.26) satisfies $\lim_{\varkappa \to \infty} v(\varkappa) = 0$.

Stability Analysis The functions Γ and $\pounds$ satisfy the locally Lipschitz condition, H (from Theorem 2.2) and R (from Theorem 2.6) exist. There exist constants $P, k > 0$ such that for all $\varkappa$:

$$|\Gamma(v(\varkappa)+u^*(\varkappa)) - \Gamma(u^*(\varkappa))| \leq P|v(\varkappa)|,$$
$$|\pounds(v(\varkappa)+u^*(\varkappa)) - \pounds(u^*(\varkappa))| \leq k|v(\varkappa)|.$$

The function φ satisfies the integral equation:

$$\begin{aligned}\varphi(\varkappa) =& \Big(\psi(0) - r(0)\Big(\pounds(\psi(-\varsigma(0)) + u^*(-\varsigma(0))) - \pounds(u^*(-\varsigma(0)))\Big)\Big)e^{-\int_0^{\varkappa} a(u)du} \\ &+ r(\varkappa)\Big(\pounds(\varphi(\varkappa-\varsigma(\varkappa)) + u^*(\varkappa-\varsigma(\varkappa))) - \pounds(u^*(\varkappa-\varsigma(\varkappa)))\Big) \\ &+ \int_0^{\varkappa} e^{-\int_s^{\varkappa} a(u)du}\Big[-\Big(2a(s)r(s)+r'(s)\Big)\Big(\pounds(\varphi(s-\varsigma(s)) \\ &\quad + u^*(s-\varsigma(s))) - \pounds(u^*(s-\varsigma(s)))\Big) - c(s)\Big(\Gamma(\varphi(s)+u^*(s)) - \Gamma(u^*(s))\Big) \\ &\quad + c(s)r(s)\Big(\Gamma(\varphi(s-\varsigma(s)) + u^*(s-\varsigma(s))) - \Gamma(u^*(s-\varsigma(s)))\Big)\Big]ds.\end{aligned} \tag{2.30}$$

Lemma 2.4. *The following norm inequality holds:*

$$\|\varphi\| \leq (1+k\|r\|)\|\psi\| + k\|r\|\|\varphi\| + \delta\Big[k(2\|a\|\|r\|+\beta)\|\varphi\| + \|c\|(1+\|r\|)P\|\varphi\|\Big]. \tag{2.31}$$

Equivalently,

$$\Big[1 - k\|r\| - \delta k(2\|a\|\|r\|+\beta) - \delta\|c\|(1+\|r\|)P\Big]\|\varphi\| \leq (1+k\|r\|)\|\psi\|. \tag{2.32}$$

Theorem 2.7 (Stability of the Zero Solution)**.** *if P and k satisfy:*

$$1 - k\|r\| - \delta k(2\|a\|\|r\|+\beta) - \delta\|c\|(1+\|r\|)P > 0, \tag{2.33}$$

then for every $\epsilon > 0$, there exists $\sigma > 0$ such that:

$$\|\psi\| < \sigma \implies |\varphi(\varkappa)| < \epsilon \quad \forall \varkappa \geq m(0).$$

That is, the zero solution of (2.26) satisfies is stable.

2.4 Application

Consider a nonlinear load consisting of a r-n junction diode, a common component in integrated electronic circuits. The voltage-current (V-I) characteristic of an ideal diode is given by:

$$i(u) = I_0\left(e^{\frac{u}{\alpha k T_0}} - 1\right), \tag{2.34}$$

where:

- T_0 is the temperature (in Kelvin)
- α is an adjustment factor ($\alpha = 1$ for germanium, $\alpha \approx 2$ for silicon)
- $k = 8.620 \times 10^{-5}$ eV/K (Boltzmann's constant)
- I_0 is the reverse saturation current

The temperature-dependent I_0 follows:

$$I_0 = K_0 T_0^{2+\eta} e^{-V_{\pounds}/(\alpha k T_0)}, \tag{2.35}$$

where:

- K_0 is a proportionality constant
- $V_{\pounds}$ is the energy gap (0.67 eV for germanium, 1.11 eV for silicon)
- η is a correction factor ($\eta = 1$ for germanium, $\eta = 0.75$ for silicon)

Silicon Diode Case Study For a silicon diode at $T_0 = 300$ K with $I_0 = 10^{-8}$ A, the characteristic becomes:

$$i(u) = 10^{-8}\left(e^{19.33u} - 1\right). \tag{2.36}$$

The system parameters are:

- $r = \dfrac{Z_0 - R_0}{A^2(Z_0 + R_0)}$
- $\dfrac{c}{a} = \dfrac{1}{Z_0} \geq 0.005$ (typical $Z_0 \leq 200\,\Omega$)

Existence of Periodic Solutions The nonlinearity satisfies:

$$\lim_{u \to 0^+} \frac{\Gamma(u)}{u} = 1.933 \times 10^{-7} < \frac{1}{Z_0}. \tag{2.37}$$

Thus, there exists $H > 0$ satisfying condition (2.3). When r and $q(\varkappa)$ satisfy (2.4), Theorem 2.1 guarantees the existence of a T-periodic solution for Eq. (2.1). The asymptotic stability of the periodic solution can be determined using:

- Theorem 2.4 for convergence to zero
- Theorem 2.5 for stability conditions

Chapter 3

Investigation of the periodicity and stability in the delay functional neutral systems

This study investigates the stability and existence of periodic solutions in neutral differential systems with time delays and variable coefficients. Utilizing Krasnoselskii's fixed point theorem, we demonstrate a set of sufficient conditions ensuring the existences of such a periodic solution. This involves transforming the system into an equivalent integral form before applying the fundamental matrix solutions alongside Floquet theory. In addition, we will analyze the asymptotic stability of these solutions, thus establishing new conditions that can ensure stability. The practical relevance of our theoretical results is supported through numerical examples, validating the proposed approach, and highlighting its suitability in areas such as electrical circuits, control systems, and biological modeling. The present study extends previous work and thereby offers a detailed framework intended for use in studying neutral differential systems where time delays are present.

keywords Mathematical Model; Neutral Differential Systems; Periodic Solutions; Asymptotic Stability; Fixed Point Method

3.1 Introduction

In 2020, Guerfi and Ardjouni [20] examined a neutral differential system with constant delay:

$$\begin{aligned} &\frac{d}{d\varkappa}u(\varkappa) - r\frac{d}{d\varkappa}u(\varkappa-\varsigma) \\ &= \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa) + \mathcal{N}(\varkappa)ru(\varkappa-\varsigma) - c\Gamma(u(\varkappa)) + cr\Gamma(u(\varkappa-\varsigma)). \end{aligned} \tag{3.1}$$

By using the fundamental matrix solution and Floquet theory, the authors converted the differential system into an integral system, making it suitable for the application of Krasnoselskii's theory. This approach provided a more comprehensive framework for analyzing periodic solutions in neutral systems.

Here, see [10] we analyze the following general neutral differential system, focusing on the asymptotic stability with the existence of its periodic solution:

$$\begin{aligned} &\frac{d}{d\varkappa}u(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\pounds(u(\varkappa-\varsigma(\varkappa))) = \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa) \\ &+ \mathcal{N}(\varkappa)r(\varkappa)\pounds(u(\varkappa-\varsigma(\varkappa))) - c(\varkappa)\Gamma(u(\varkappa)) + c(\varkappa)r(\varkappa)\Gamma(u(\varkappa-\varsigma(\varkappa))), \end{aligned} \tag{3.2}$$

here ς is a positive differentiable function, c and r are continuously and twic continuously differentiable. Moreover, the non-singular matrix i.e., $n \times n$ regard to continuous real value function is denoted by $\mathcal{N}$,$\mathcal{Q}$, $\pounds$, and Γ are assumed to be continuously differentiable functions.

The Krasnoselskii's fixed point theory is used on system (3.2) to demonstrate the sufficient condition for the stability and existence of periodic solution. This work extends and generalizes the findings of Ding, Li, Mansouri, Ardjouni, Djoudi, Guerfi, providing a more comprehensive framework for analyzing neutral differential systems with variable coefficients and delays.

The work is structured as follows: Section 2 explores the existence of periodic solution by transforming the neutral differential system into an equivalent integral system using the fundamental matrix solution and Floquet theory. Section 3 discusses the asymptotic-stability of equilibrium and periodic solution by applying Krasnoselskii's theorem to confirm the asymptotic-stability of trivial solution. Additionally, we present sufficient condition to ensure the stability of non-constant periodic solution. Practical examples are included to analyze the utility of the theoretical result, with particular emphasis on applications in electrical engineering and control systems.

The overall goal of this study is to further the understanding of transmission line problems, especially in how these problems relate to neutral differential equations and systems. As such, we provide new conditions intended to guarantee the stability and existence of periodic solution. These condition is more general and applicable to a wider range of systems than those previously established in the literature. It is anticipated that these findings will help to improve the understanding of dynamical systems with time delays, enhancing their applicability in a variety of scenarios across both science and engineering.

3.2 Existence of periodic solutions

3.2.1 Existence results

Throughout this section, we use $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$ to represent the spaces of continuous differentiable and continuous function $\vartheta : \mathbb{R} \to \mathbb{R}^n$. For any $0 < T$, we define

$$\mathcal{C}_T = \{\vartheta \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n) \mid \vartheta(\varkappa + T) = \vartheta(\varkappa)\},$$

Under the supremum norm which forms a Banach space:

$$\|\vartheta\|_0 = \sup_{\varkappa \in \mathbb{R}} |\vartheta(\varkappa)| = \sup_{\varkappa \in [0,T]} |\vartheta(\varkappa)|,$$

For $x \in \mathbb{R}^n$, in which $|.|$ indicates the infinite-norm. Furthermore, we denote $\mathcal{C}_T \cap \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ as $\mathcal{C}_T^1$, which constitutes a Banach space equipped with the given norm

$$\|\vartheta\|_1 = \|\vartheta\|_0 + \|\vartheta'\|_0,$$

over one period interval. For $n \times n$ real matrix $\mathcal{N}$ with the norm $|\mathcal{N}|_* = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$ and $\|\mathcal{N}\|_* = \sup_{\varkappa \in [0,T]} |\mathcal{N}|_*$. It is also consider that $\mathcal{N}(\varkappa + T) = \mathcal{N}(\varkappa)$, $\mathcal{Q}(\varkappa + T) = \mathcal{Q}(\varkappa)$, $r(\varkappa + T) = r(\varkappa)$, $c(\varkappa + T) = c(\varkappa)$ and $\varsigma(\varkappa + T) = \varsigma(\varkappa)$. In addition, the function $\pounds$ is assumed to demonstrate global Lipschitz continuity, requiring that one can determine a positive constant l that follows the condition $|\pounds(x) - \pounds(y)| \leq l\,|x - y|$, implying that $\left\|\pounds'\right\|_0 = \gamma_1$.

To facilitate analysis, it can be assumed that $\mathbb{L}$ denotes a positive constant exhibiting a sufficiently small value. Given this assumption, the system (3.2) can be rewritten as

$$\begin{aligned} &\frac{d}{d\varkappa}v(\varkappa) - r_1(\varkappa)\frac{d}{d\varkappa}\pounds(v(\varkappa - \wp(\varkappa)) \\ &= \mathbb{L}\mathcal{Q}_1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)r_1(\varkappa)\pounds(v(\varkappa - \wp(\varkappa)) \\ &- \mathbb{L}c_1(\varkappa)\Gamma(v(\varkappa)) + \mathbb{L}c_1(\varkappa)r_1(\varkappa)\Gamma(v(\varkappa - \wp(\varkappa))), \end{aligned} \tag{3.3}$$

here $v(\varkappa) = u(\mathbb{L}\varkappa)$, $\wp(\varkappa) = \dfrac{\varsigma(\mathbb{L}\varkappa)}{\mathbb{L}}$, $r_1(\varkappa) = r(\mathbb{L}\varkappa)$, $\mathcal{Q}_1(\varkappa) = \mathcal{Q}(\mathbb{L}\varkappa)$, $\mathcal{N}_1(\varkappa) = \mathcal{N}(\mathbb{L}\varkappa)$, $c_1(\varkappa) = c(\mathbb{L}\varkappa)$ and $\ell = \dfrac{T}{L}$.

To begin, we start with the following definition:

Definition 3.1. *Let $\mathcal{N}_1$ denote a periodic matrix where the period is given by $\ell = \dfrac{T}{\mathbb{L}}$. The below*

$$y'(\varkappa) = \mathbb{L}\mathcal{N}_1(\varkappa)y(\varkappa), \tag{3.4}$$

is called non-critical with regard to ℓ. In the trivial case $y = 0$, a periodic solution exits with a period ℓ.

In all this work, we considered that the given system (3.4) satisfies the noncritical condition. A few known properties are summarized from [18] below. For this, we define the basic matrix Υ of (3.4), with $\Upsilon(0) = I$ with I denoting the identity matrix $n \times n$. With these definitions:

(a) $det\Upsilon(\varkappa) \neq 0$.

(b) According to Floquet theory, a matrix B must exist that satisfies $\Upsilon(\varkappa + \ell) = \Upsilon(\varkappa)e^{B\ell}$.

(c) System (3.4) satisfies the noncritical condition $\iff$ $det(I - \Upsilon(\ell)) \neq 0$.

Lemma 3.1 ([20]). *Let us take the given below equation*

$$y'(\varkappa) = \mathbb{L}\mathcal{N}_1(\varkappa)\, y(\varkappa) + g(\varkappa). \tag{3.5}$$

Given that $g \in C_\ell$, and that the matrix $\mathbb{L}\mathcal{N}_1$ has a period of ℓ, then

$$y(\varkappa) = \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell) - I\right)^{-1} \int_{\varkappa}^{\varkappa+\ell} \Upsilon^{-1}(s)\, g(s)\, ds,$$

is a unique ℓ - periodic solution of (3.5).

Proof. Since $K(\varkappa)\Upsilon^{-1}(\varkappa) = I$, it follows that

$$\begin{aligned} 0 &= \frac{d}{d\varkappa}\left[\Upsilon(\varkappa)\Upsilon^{-1}(\varkappa)\right] \\ &= \frac{d}{d\varkappa}(\Upsilon(\varkappa))\Upsilon^{-1}(\varkappa) + \Upsilon(\varkappa)\frac{d}{d\varkappa}\left[\Upsilon^{-1}(\varkappa)\right] \\ &= (\mathbb{L}\mathcal{N}_1(\varkappa)\Upsilon(\varkappa))\Upsilon^{-1}(\varkappa) + \Upsilon(\varkappa)\frac{d}{d\varkappa}\left[\Upsilon^{-1}(\varkappa)\right] \\ &= \mathbb{L}\mathcal{N}_1(\varkappa) + \Upsilon(\varkappa)\frac{d}{d\varkappa}\left[\Upsilon^{-1}(\varkappa)\right]. \end{aligned}$$

This implies

$$\frac{d}{d\varkappa}\left[\Upsilon^{-1}(\varkappa)\right] = -\Upsilon^{-1}(\varkappa)\mathbb{L}\mathcal{N}_1(\varkappa). \tag{3.6}$$

If x is a solution of (3.5) with $y(0)=y_0$, then

$$\begin{aligned}\frac{d}{d\varkappa}\left[\Upsilon^{-1}(\varkappa)y(\varkappa)\right] &= \frac{d}{d\varkappa}\left[\Upsilon^{-1}(\varkappa)\right]y(\varkappa)+\Upsilon^{-1}(\varkappa)\frac{d}{d\varkappa}y(\varkappa)\\ &= -\Upsilon^{-1}(\varkappa)\mathbb{L}\mathcal{N}_1(\varkappa)y(\varkappa)+\Upsilon^{-1}(\varkappa)\left[\mathbb{L}\mathcal{N}_1(\varkappa)y(\varkappa)+g(\varkappa)\right]\\ &= \Upsilon^{-1}(\varkappa)g(\varkappa),\end{aligned}$$

by (3.6). An integration of the above equation from 0 to $\varkappa$ yields

$$y(\varkappa)=\Upsilon(\varkappa)y(0)+\Upsilon(\varkappa)\int_0^{\varkappa}\Upsilon^{-1}(s)g(s)ds. \tag{3.7}$$

Since $y(\ell)=y_0=y(0)$, we get

$$y(0)=(I-\Upsilon(\ell))^{-1}\int_0^{\ell}\Upsilon(\ell)\Upsilon^{-1}(s)g(s)ds. \tag{3.8}$$

A substitution of (3.8) into (3.7) yields

$$y(\varkappa)=\Upsilon(\varkappa)(I-\Upsilon(\ell))^{-1}\int_0^{\ell}\Upsilon(\ell)\Upsilon^{-1}(s)g(s)ds+\Upsilon(\varkappa)\int_0^{\varkappa}\Upsilon^{-1}(s)g(s)ds. \tag{3.9}$$

Since

$$(I-\Upsilon(\ell))^{-1}=\left[\Upsilon(\ell)\left(\Upsilon^{-1}(\ell)-I\right)\right]^{-1}=\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\Upsilon^{-1}(\ell),$$

(3.9) becomes

$$\begin{aligned}y(\varkappa) &= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\int_0^{\ell}\Upsilon^{-1}(s)g(s)ds+\Upsilon(\varkappa)\int_0^{\varkappa}\Upsilon^{-1}(s)g(s)ds\\ &= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\left[\int_0^{\ell}\Upsilon^{-1}(s)g(s)ds+\Upsilon^{-1}(\ell)\int_0^{\varkappa}\Upsilon^{-1}(s)g(s)ds-\int_0^{\varkappa}\Upsilon^{-1}(s)g(s)ds\right]\\ &= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\left[\int_{\varkappa}^{\ell}\Upsilon^{-1}(s)g(s)ds+\Upsilon^{-1}(\ell)\int_0^{\varkappa}\Upsilon^{-1}(s)g(s)ds\right].\end{aligned}$$

By letting $s=\mu-\ell$, the above expression implies

$$\begin{aligned}&y(\varkappa)\\ &= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\left\{\int_{\varkappa}^{\ell}\Upsilon^{-1}(s)g(s)ds+\Upsilon^{-1}(\ell)\int_{\ell}^{\varkappa+\ell}\Upsilon^{-1}(\mu-\ell)g(\mu-\ell)d\mu\right\}.\end{aligned} \tag{3.10}$$

By (b) we have $\Upsilon(\varkappa-\ell)=\Upsilon(\varkappa)e^{-B\ell}$ and $\Upsilon(\ell)=e^{B\ell}$. Hence,

$$\Upsilon^{-1}(\ell)\Upsilon^{-1}(\mu-\ell)=\Upsilon^{-1}(\mu).$$

Consequently, (3.10) becomes

$$y(\varkappa)=\Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\left\{\int_{\varkappa}^{\ell}\Upsilon^{-1}(s)g(s)ds+\int_{\ell}^{\varkappa+\ell}\Upsilon^{-1}(s)g(s)ds\right\}.$$

□

Theorem 3.1 (**Krasnoselskii's fixed point theorem** [13]). *We define* $\mathbb{D}$ *to represent a nonempty, convex, alongside closed subset, within a Banach space* $(\mathbb{E}, \|.\|)$. *Here, mappings* $\digamma$ *and* $\mathcal{S}$ *transform* $\mathbb{D}$ *into* $\mathbb{E}$, *and furthermore satisfy the following conditions:*

(i) $\digamma s + \mathcal{S}\varkappa \in \mathbb{D}, \forall \varkappa, s \in \mathbb{D}$,

(ii) $\digamma$ *exhibits continuity, and* $\digamma \mathbb{D}$ *is contain in a compact set,*

(iii) $\mathcal{S}$ *represents a contraction as characterized by a constant* $\epsilon < 1$.

Given these conditions, an $x \in \mathbb{D}$ *will satisfy* $x = \digamma x + \mathcal{S}x$.

Lemma 3.1 and Theorem 3.1 serve as the basis to ensure the existence of periodic solution of (3.2).

Theorem 3.2. *Assuming that* $\Gamma \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{Q}_1 \in \mathcal{C}^1_\ell(\mathbb{R}, \mathbb{R}^n)$, $\mathcal{N}_1 \in \mathcal{C}^1_\ell(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$, $c_1 \in \mathcal{C}^1_\ell(\mathbb{R}, \mathbb{R})$ *and* $r_1 \in \mathcal{C}^2_\ell(\mathbb{R}, \mathbb{R})$, *suppose that* $\|r_1'\| = \delta_1$, $\|r_1''\| = \delta_2$ *and* $\|1 - \wp'(\varkappa)\| = \gamma_2$. *If a constant* $\rho \in [0, 1]$ *and* $D > 0$ *exist such that:*

$$l\left(\|r_1\|\left(1 + \gamma_2 D\right) + \delta_1\right) \leq \rho, \tag{3.11}$$

$$\eta_2 \delta_1 \left(1 + \eta_1\right) < D, \tag{3.12}$$

$$\sup_{\|u\| \leq D} |\Gamma(u)| < \frac{D - \eta_2 \delta_1 \left(1 + \eta_1\right)}{\mathbb{L}\eta_1 \|c_1\|}, \tag{3.13}$$

$$\|r_1\| < \frac{D - \eta_2 \delta_1 \left(1 + \eta_1\right) - \mathbb{L}\eta_1 \|c_1\| \sup_{\|u\| \leq D} |\Gamma(u)|}{\gamma_1 \gamma_2 D + \eta_2 \left(1 + 2\mathbb{L}\eta_1 \|\mathcal{N}_1\|_*\right) + \mathbb{L}\eta_1 \|c_1\| \sup_{\|u\| \leq D} |\Gamma(u)|}, \tag{3.14}$$

and

$$\begin{aligned} \|\mathcal{Q}_1\|_0 < \Bigg[& \frac{\left(1 - \gamma_1 \gamma_2 \|r_1\|\right) D - \left(\delta_1 + \|r_1\|\right) \eta_2 - \left(2\mathbb{L} \|\mathcal{N}_1\|_* \|r_1\| + \delta_1\right) \eta_1 \eta_2}{\mathbb{L}\eta_1} \\ & - \frac{\mathbb{L}\eta_1 \|c_1\| \left(1 + \|r_1\|\right) \sup_{\|u\| \leq D} |\Gamma(u)|}{\mathbb{L}\eta_1} \Bigg], \end{aligned} \tag{3.15}$$

where $\eta_1 = 1 + \left(1 + \mathbb{L} \|\mathcal{N}_1\|_*\right) \mu \ell$, $\eta_2 = lD + |\pounds(0)|$ *and*

$$\mu = \sup_{\varkappa \in [0, \ell]} \left(\sup_{\varkappa \leq s \leq \varkappa + \ell} \left| \left[\Upsilon(s) \left(\Upsilon^{-1}(\ell) - I \right) \Upsilon^{-1}(\varkappa) \right]^{-1} \right|_* \right).$$

This implies that there is a T*-periodic solution of (3.2).*

Proof. By the condition (3.15), we obtain

$$
\begin{aligned}
&\mathbb{L}\eta_1 \|\mathcal{Q}_1\|_0 + \gamma_1\gamma_2 \|r_1\| D + (\delta_1 + \|r_1\|)\,\eta_2 + (2\mathbb{L}\,\|\mathcal{N}_1\|_* \|r_1\| + \delta_1)\,\eta_1\eta_2 \\
&+ \mathbb{L}\eta_1 \|c_1\| (1 + \|r_1\|) \sup_{\|u\|\le D} |\Gamma(u)| \\
&\le \mathbb{L}\eta_1 [\frac{(1-\gamma_1\gamma_2 \|r_1\|) D - (\delta_1 + \|r_1\|)\,\eta_2 - (2\mathbb{L}\,\|\mathcal{N}_1\|_* \|r_1\| + \delta_1)\,\eta_1\eta_2}{L\eta_1} \\
&- \frac{\mathbb{L}\eta_1 \|c_1\| (1 + \|r_1\|) \sup_{\|u\|\le D} |\Gamma(u)|}{\mathbb{L}\eta_1}] + \gamma_1\gamma_2 \|r_1\| D + (\delta_1 + \|r_1\|)\,\eta_2 \\
&+ (2\mathbb{L}\,\|\mathcal{N}_1\|_* \|r_1\| + \delta_1)\,\eta_1\eta_2 + \mathbb{L}\eta_1 \|c_1\| (1 + \|r_1\|) \sup_{\|u\|\le D} |\Gamma(u)| \\
&= D. \qquad (3.16)
\end{aligned}
$$

We will show that there exists a ℓ-periodic solution of (3.3). Assume

$$
\mathbb{S} = \left\{ \vartheta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n), \|\vartheta\|_1 = \|\vartheta\|_0 + \left\|\vartheta'\right\|_0 < +\infty \right\},
$$

and assume the following bounded and closed convex set

$$
\mathbb{M} = \left\{ \vartheta \in \mathcal{C}_\ell^1, \|\vartheta\|_1 < D \right\},
$$

of the Banach space $\mathbb{S}$. For every $\vartheta \in \mathbb{M}$, suppose we have the system

$$
\begin{aligned}
\frac{d}{d\varkappa} v(\varkappa) &= \mathbb{L}\mathcal{N}_1(\varkappa) v(\varkappa) + \mathbb{L}\mathcal{Q}_1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa) r_1(\varkappa) \pounds(v(\varkappa - \wp(\varkappa))) \\
&- \mathbb{L}c_1(\varkappa)\Gamma(v(\varkappa)) + \mathbb{L}c_1(\varkappa) r_1(\varkappa)\Gamma(v(\varkappa - \wp(\varkappa))) - r_1(\varkappa)\frac{d}{d\varkappa}\pounds(v(\varkappa - \wp(\varkappa))) \\
&= \mathbb{L}\mathcal{N}_1(\varkappa) v(\varkappa) + g(\varkappa),
\end{aligned}
$$

with

$$
\begin{aligned}
g(\varkappa) &= \mathbb{L}\mathcal{Q}_1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa) r_1(\varkappa) \mathbb{L}(v(\varkappa - \wp(\varkappa))) - \mathbb{L}c_1(\varkappa)\Gamma(v(\varkappa)) \\
&+ \mathbb{L}c_1(\varkappa) r_1(\varkappa)\Gamma(v(\varkappa - \wp(\varkappa))) - r_1(\varkappa)\frac{d}{d\varkappa}\pounds(v(\varkappa - \wp(\varkappa))).
\end{aligned}
$$

By using Lemma 3.1, there is a unique ℓ-periodic solution of this system of the form:

$$
\begin{aligned}
v(\varkappa) = \mathrm{Y}(\varkappa)\left(\mathrm{Y}^{-1}(\ell) - I\right)^{-1} \int_{\varkappa}^{\varkappa+\ell} \mathrm{Y}^{-1}(s)\,[\mathbb{L}\mathcal{Q}_1(s) + \mathbb{L}\mathcal{N}_1(s) r_1(s) \pounds(v(s - \wp(s))) \\
-\mathbb{L}c_1(s)\Gamma(v(s)) + \mathbb{L}c_1(s) r_1(s)\Gamma(v(s - \wp(s))) - r_1(s)\frac{\partial}{\partial s}\pounds(v(s - \wp(s)))\Big]\, ds. \qquad (3.17)
\end{aligned}
$$

As $\mathrm{Y}^{-1}(\varkappa + \ell) = e^{-B\ell}\mathrm{Y}^{-1}(\varkappa)$, this implies that

$$
\mathrm{Y}^{-1}(\varkappa + \ell) - \mathrm{Y}^{-1}(\varkappa) = e^{-B\ell}\mathrm{Y}^{-1}(\varkappa) - \mathrm{Y}^{-1}(\varkappa) = \left(\mathrm{Y}^{-1}(\ell) - I\right)\mathrm{Y}^{-1}(\varkappa). \qquad (3.18)
$$

Also

$$\frac{d}{d\varkappa}\Upsilon^{-1}(\varkappa) = -\Upsilon^{-1}(\varkappa)\mathbb{L}\mathcal{N}_1(\varkappa), \tag{3.19}$$

now, (3.18) and (3.19) into (3.17) implies

$$\begin{aligned}
&\Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\int_{\varkappa}^{\varkappa+\ell}\Upsilon^{-1}(s)\,r_1(s)\frac{\partial}{\partial s}\pounds(v(s-\wp(s)))ds \\
&= r_1(\varkappa)\pounds(v(\varkappa-\wp(\varkappa))) - \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\int_{\varkappa}^{\varkappa+\ell}\Big[\Upsilon^{-1}(s)\frac{\partial}{\partial s}r_1(s) \\
&-r_1(s)\Upsilon^{-1}(s)\,\mathbb{L}\mathcal{N}_1(s)\Big]\,\pounds(v(s-\wp(s)))ds \\
&= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\int_{\varkappa}^{\varkappa+\ell}\Upsilon^{-1}(s)\left[L\mathcal{N}_1(s)q_1(s)-q_1'(s)\right]\pounds(v(s-\wp(s)))ds \\
&+r_1(\varkappa)\pounds(v(\varkappa-\wp(\varkappa))).
\end{aligned}$$

Therefore

$$\begin{aligned}
v(\varkappa) = \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}\int_{\varkappa}^{\varkappa+\ell}\Upsilon^{-1}(s)\Big[&\mathbb{L}\mathcal{Q}_1(s) \\
&+\left(2\mathbb{L}\mathcal{N}_1(s)r_1(s)-r_1'(s)\right)\pounds(v(s-\wp(s))) \\
&-\mathbb{L}c_1(s)\Gamma(v(s))+\mathbb{L}c_1(s)r_1(s)\Gamma(v(s-\wp(s)))\Big]ds \\
&+r_1(\varkappa)\pounds(v(\varkappa-\wp(\varkappa))).
\end{aligned}$$

Here, we introduce $\mathcal{A}_1$ and $\mathcal{B}_1$ operators as

$$\begin{aligned}
\left(\mathcal{A}_1\vartheta\right)(\varkappa) = \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}&\int_{\varkappa}^{\varkappa+\ell}\Upsilon^{-1}(s)\left[\mathbb{L}\mathcal{Q}_1(s)+(2\mathbb{L}\mathcal{N}_1(s)r_1(s)\right. \\
&\left.-r_1'(s))\pounds(\vartheta(s-\wp(s)))-\mathbb{L}c_1(s)\Gamma(\vartheta(s))+\mathbb{L}c_1(s)r_1(s)\Gamma(\vartheta(s-\wp(s)))\right]ds,
\end{aligned}$$

and

$$\left(\mathcal{B}_1\vartheta\right)(\varkappa) = r_1(\varkappa)\pounds(v(\varkappa-\wp(\varkappa))).$$

To establish the existence of an ℓ-periodic solution for (3.3), it is necessary to verify that $\mathcal{A}_1$ and $\mathcal{B}_1$ fulfill the conditions of Theorem 3.1. We define $x(\varkappa+\ell)=x(\varkappa)$, $y(\varkappa+\ell)=y(\varkappa)$ and $\|x\|_1 \le D$, $\|y\|_1 \le D$ for $x,y\in\mathbb{M}$. Next, we will investigate $\mathcal{A}_1x+\mathcal{B}_1y$. We have

$$\begin{aligned}
\left(\mathcal{A}_1x\right)(\varkappa+\ell) = \Upsilon(\varkappa+\ell)\left(\Upsilon^{-1}(\ell)-I\right)^{-1}&\int_{\varkappa+\ell}^{\varkappa+2\ell}\Upsilon^{-1}(s)\left[\mathbb{L}\mathcal{Q}_1(s)+(2\mathbb{L}\mathcal{N}_1(s)r_1(s)\right. \\
&\left.-r_1'(s))\pounds(x(s-\wp(s)))-\mathbb{L}c_1(s)\Gamma(x(s))+\mathbb{L}c_1(s)r_1(s)\Gamma(x(s-\wp(s)))\right]ds.
\end{aligned}$$

Taking $s = \mu + \ell$, then, we have the following from the above:

$$\begin{aligned}(\mathcal{A}_1 x)(\varkappa + \ell) &= \Upsilon(\varkappa + \ell)\left(\Upsilon^{-1}(\ell) - I\right)^{-1} \int_{\varkappa+\ell}^{\varkappa+2\ell} \Upsilon^{-1}(\mu + \ell)\,[\mathbb{L}\mathcal{Q}_1(\mu + \ell) \\ &+ (2\mathbb{L}\mathcal{N}_1(\mu+\ell) r_1(\mu+\ell) - r_1'(\mu+\ell))\pounds(x(\mu+\ell-\wp(\mu+\ell))) \\ &- \mathbb{L}c_1(\mu+\ell)\Gamma(x(\mu+\ell)) + \mathbb{L}c_1(\mu+\ell) r_1(\mu+\ell)\Gamma(x(\mu+\ell-\wp(\mu+\ell)))d\mu] \\ &= \Upsilon(\varkappa+\ell)\left(\Upsilon^{-1}(\ell) - I\right)^{-1} \int_{\varkappa}^{\varkappa+\ell} \Upsilon^{-1}(\mu+\ell)\,[\mathbb{L}\mathcal{Q}_1(\mu) + (2\mathbb{L}\mathcal{N}_1(\mu) r_1(\mu) \\ &- r_1'(\mu))\pounds(x(\mu-\wp(\mu))) - \mathbb{L}c_1(\mu)\Gamma(x(\mu)) + \mathbb{L}c_1(\mu) r_1(\mu)\Gamma(x(\mu-\wp(\mu)))\Big]\,d\mu.\end{aligned}$$

Here, through (b), we obtained

$$\begin{aligned}&\Upsilon(\varkappa+\ell)\left(\Upsilon^{-1}(\ell) - I\right)^{-1}\Upsilon^{-1}(\mu+\ell) \\ &= \Upsilon(\varkappa+\ell)\left(\Upsilon^{-1}(\ell) - I\right)^{-1} e^{-B\ell}\Upsilon^{-1}(\mu) \\ &= \Upsilon(\varkappa+\ell)\left[e^{B\ell}\left(\Upsilon^{-1}(\ell) - I\right)\right]^{-1}\Upsilon^{-1}(\mu) \\ &= \Upsilon(\varkappa)\, e^{B\ell}\,(I - \Upsilon(\ell))^{-1}\Upsilon^{-1}(\mu) \\ &= \Upsilon(\varkappa)\left[(I - \Upsilon(\ell))\, e^{-B\ell}\right]^{-1}\Upsilon^{-1}(\mu) \\ &= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell) - I\right)^{-1}\Upsilon^{-1}(\mu).\end{aligned}$$

As a result of the above, we have the following

$$\begin{aligned}(\mathcal{A}_1 x)(\varkappa+\ell) &= \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell) - I\right)^{-1}\int_{\varkappa}^{\varkappa+\ell}\Upsilon^{-1}(s)\,[\mathbb{L}\mathcal{Q}_1(s) + (2\mathbb{L}\mathcal{N}_1(s) r_1(s) \\ &- r_1'(s))\pounds(x(s-\wp(s))) - \mathbb{L}c_1(s)\Gamma(x(s)) + \mathbb{L}c_1(s) r_1(s)\Gamma(x(s-\wp(s)))\Big]\,ds \\ &= (\mathcal{A}_1 x)(\varkappa),\end{aligned}$$

and

$$\begin{aligned}(\mathcal{B}_1 y)(\varkappa+\ell) &= r_1(\varkappa+\ell)\pounds(y(\varkappa+\ell-\wp(\varkappa+\ell))) \\ &= r_1(\varkappa)\pounds(y(\varkappa-\wp(\varkappa))) \\ &= (\mathcal{B}_1 y)(\varkappa),\end{aligned}$$

thus $(\mathcal{A}_1 x + \mathcal{B}_1 y)(\varkappa+\ell) = (\mathcal{A}_1 x + \mathcal{B}_1 y)(\varkappa)$. We also obtained

$$\begin{aligned}(\mathcal{A}_1 x)'(\varkappa) &= \Upsilon'(\varkappa)\left(\Upsilon^{-1}(\ell) - I\right)^{-1}\int_{\varkappa}^{\varkappa+\ell}\Upsilon^{-1}(s)\,[\mathbb{L}\mathcal{Q}1(s) + (2\mathbb{L}\mathcal{N}_1(s) r_1(s) \\ &- r_1'(s))\pounds(x(s-\wp(s))) - \mathbb{L}c_1(s)\Gamma(x(s)) + \mathbb{L}c_1 r_1(s)\Gamma(x(s-\wp(s)))\Big]\,ds \\ &+ \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell) - I\right)^{-1}\left[\Upsilon^{-1}(\varkappa+\ell) - \Upsilon^{-1}(\varkappa)\right][\mathbb{L}\mathcal{Q}1(\varkappa) \\ &+ \Big(2\mathbb{L}\mathcal{N}_1(\varkappa) r_1(\varkappa) - r_1'(\varkappa)\Big)\pounds(x(\varkappa-\wp(\varkappa))) - \mathbb{L}c_1(\varkappa)\Gamma(x(\varkappa)) \\ &+ \mathbb{L}c_1(\varkappa) r_1(\varkappa)\Gamma(x(\varkappa-\wp(\varkappa)))].\end{aligned} \tag{3.20}$$

Since

$$\mathrm{Y}'(\varkappa)=\mathbb{L}\mathcal{N}_1(\varkappa)\mathrm{Y}(\varkappa), \tag{3.21}$$

with $\mathrm{Y}^{-1}(\varkappa+\ell)=e^{-B\ell}\mathrm{Y}^{-1}(\varkappa)$ and

$$\mathrm{Y}^{-1}(\varkappa+\ell)-\mathrm{Y}^{-1}(\varkappa)=e^{-B\ell}\mathrm{Y}^{-1}(\varkappa)-\mathrm{Y}^{-1}(\varkappa)=\Big(\mathrm{Y}^{-1}(\ell)-I\Big)\mathrm{Y}^{-1}(\varkappa). \tag{3.22}$$

From (3.20), (3.21) and (3.22), we have

$$\begin{aligned}(\mathcal{A}_1x)'(\varkappa) &= \mathbb{L}\mathcal{N}_1(\varkappa)(\mathcal{A}_1x)(\varkappa)+\mathbb{L}\mathcal{Q}1(\varkappa)\\ &\quad+\Big(2\mathbb{L}\mathcal{N}_1(\varkappa)r_1(\varkappa)-r_1'(\varkappa)\Big)\pounds(x(\varkappa-\wp(\varkappa)))-\mathbb{L}c_1(\varkappa)\Gamma(x(\varkappa))\\ &\quad+\mathbb{L}c_1(\varkappa)r_1(\varkappa)\Gamma(x(\varkappa-\wp(\varkappa)))|\,.\end{aligned}$$

Thus, $\|\mathcal{A}_1x\|_1=\|\mathcal{A}_1x\|_0+\left\|(\mathcal{A}_1x)'\right\|_0$ with

$$\begin{aligned}\|\mathcal{A}_1x\|_0 &= \sup_{\varkappa\in[0,\ell]}\left|\mathrm{Y}(\varkappa)\Big(\mathrm{Y}^{-1}(\ell)-I\Big)^{-1}\int_{\varkappa}^{\varkappa+\ell}\mathrm{Y}^{-1}(s)\Big[\mathbb{L}\mathcal{Q}_1(s)+\Big(2LM_1(t)q_1(t)-q_1'(t)\Big)\right.\\ &\quad\times g(x(t-\tau(t)))-Lb_1(t)f(x(t))+Lb_1(t)q_1(t)f(x(t-\tau(t)))\Big]\,ds|\\ &\le\sigma\omega\left[L\|P_1\|_0+(2L\|M_1\|_*\|q_1\|+\beta_1)\,\theta_2+L\|b_1\|(1+\|q_1\|)\sup_{\|u\|\le H}|f(u)|\right].\end{aligned}$$

Where $\eta_2=lD+|\pounds(0)|$ and

$$\begin{aligned}\left\|(\mathcal{A}_1x)'\right\|_0 &= \sup_{\varkappa\in[0,\ell]}|\mathbb{L}\mathcal{N}_1(\varkappa)(\mathcal{A}_1x)(\varkappa)+\mathbb{L}\mathcal{Q}1(\varkappa)\\ &\quad+\Big(2\mathbb{L}\mathcal{N}_1(\varkappa)r_1(\varkappa)-r_1'(\varkappa)\Big)\pounds(x(\varkappa-\wp(\varkappa)))-\mathbb{L}c_1(\varkappa)\Gamma(x(\varkappa))\\ &\quad+\mathbb{L}c_1(\varkappa)r_1(\varkappa)\Gamma(x(\varkappa-\wp(\varkappa)))|\\ &\le \mathbb{L}\|\mathcal{N}_1\|_*\|\mathcal{A}_1x\|_0+\mathbb{L}\|\mathcal{Q}_1\|_0+(2\mathbb{L}\|\mathcal{N}_1\|_*\|r_1\|+\delta_1)\,\eta_2\\ &\quad+\mathbb{L}\|c_1\|(1+\|r_1\|)\sup_{\|u\|\le D}|\Gamma(u)|\,.\end{aligned}$$

Then

$$\|\mathcal{A}_1x\|_1\le\eta_1\left[\mathbb{L}\|\mathcal{Q}_1\|_0+(2\mathbb{L}\|\mathcal{N}_1\|_*\|r_1\|+\delta_1)\,\eta_2+\mathbb{L}\|c_1\|(1+\|r_1\|)\sup_{\|u\|\le D}|\Gamma(u)|\right].$$

Where $\eta_1=1+(1+\mathbb{L}\|\mathcal{N}_1\|_*)\,\mu\ell$ and

$$(\mathcal{B}_1y)'(\varkappa)=r_1'(\varkappa)\pounds(y(\varkappa-\wp(\varkappa)))+r_1(\varkappa)\Big(1-\wp'(\varkappa)\Big)y'(\varkappa-\wp(\varkappa))\pounds'(y(\varkappa-\wp(\varkappa))).$$

Thus

$$\begin{aligned} \|\mathcal{B}_1 y\|_1 &= \|\mathcal{B}_1 y\|_0 + \left\|(\mathcal{B}_1 y)'\right\|_0 \\ &\leq (\delta_1 + \|r_1\|)\,\eta_2 + \gamma_1\gamma_2\,\|r_1\|\,D. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathcal{A}_1 x + \mathcal{B}_1 y\|_1 \leq{}& \|\mathcal{A}_1 x\|_1 + \|\mathcal{B}_1 y\|_1 \\ &\leq \eta_1 \left[\mathbb{L}\,\|\mathcal{Q}_1\|_0 + (2\mathbb{L}\,\|\mathcal{N}_1\|_*\,\|r_1\| + \delta_1)\,\eta_2 + \mathbb{L}\,\|c_1\|\,(1+\|r_1\|) \sup_{\|u\|\leq D} |\Gamma(u)|\right] \\ &+ (\delta_1 + \|r_1\|)\,\eta_2 + \gamma_1\gamma_2\,\|r_1\|\,D \\ &= \mathbb{L}\eta_1\,\|\mathcal{Q}_1\|_0 + \gamma_1\gamma_2\,\|r_1\|\,D + (\delta_1 + \|r_1\|)\,\eta_2 + (2\mathbb{L}\,\|\mathcal{N}_1\|_*\,\|r_1\| + \delta_1)\,\eta_1\eta_2 \\ &+ \mathbb{L}\eta_1\,\|c_1\|\,(1+\|r_1\|) \sup_{\|u\|\leq D} |\Gamma(u)|\,. \end{aligned}$$

By (3.16), it follows that $\|\mathcal{A}_1 x + \mathcal{B}_1 y\|_1 \leq D$. Consequently, we have $\mathcal{A}_1 x + \mathcal{B}_1 y \in \mathbb{M}$.

For all $x + y \in \mathbb{M}$, the following inequalities hold:

$$\|\mathcal{A}_1 x\|_0 \leq D, \quad \text{and} \quad \left\|(\mathcal{A}_1 x)'\right\|_0 \leq D.$$

By the Ascoli Arzela Lemma, the subset $\mathcal{A}_1\mathbb{M}$ of $\mathcal{C}_\ell$ is pre-compact. Thus, for any sub-sequence $\{x_n\}$ in $\mathbb{M}$, $\exists$ a further sub-sequence $\{x_{n_k}\}$ such that $\mathcal{A}_1 x_{n_k} \to x_0 \in \mathcal{C}^1_\ell$ as $k \to +\infty$. Therefore, we obtain

$$\begin{aligned} (\mathcal{A}_1 x)''(\varkappa) ={}& \mathbb{L}\mathcal{N}_1'(\varkappa)\,(\mathcal{A}_1 x)(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)\,(\mathcal{A}_1 x)'(\varkappa) + \mathbb{L}\mathcal{Q}_1'(\varkappa) \\ &+ \left(2\mathbb{L}\mathcal{N}_1'(\varkappa) r_1(\varkappa) + 2\mathbb{L}\mathcal{N}_1(\varkappa) r_1'(\varkappa) - r_1''(\varkappa)\right) \pounds(x(\varkappa - \wp(\varkappa))) \\ &+ \left(2\mathbb{L}\mathcal{N}_1(\varkappa) r_1(\varkappa) - r_1'(\varkappa)\right)\left(1 - \wp'(\varkappa)\right) x'(\varkappa - \wp(\varkappa))\,\pounds'(x(\varkappa - \wp(\varkappa))) \\ &- \mathbb{L}\left[c_1'(\varkappa)\Gamma(x(\varkappa)) + c_1(\varkappa) x'(\varkappa)\,\Gamma'(x(\varkappa))\right] \\ &+ \mathbb{L}\left[c_1'(\varkappa) r_1(\varkappa) + c_1(\varkappa) r_1'(\varkappa)\right]\Gamma(x(\varkappa - \wp(\varkappa))) \\ &+ \mathbb{L} c_1(\varkappa) r_1(\varkappa)\left(1 - \wp'(\varkappa)\right) x'(\varkappa - \wp(\varkappa))\Gamma'(x(\varkappa - \wp(\varkappa))). \end{aligned}$$

Thus

$$\begin{aligned}
\sup_{t\in[0,\ell]}\left|(\mathcal{A}_1x)''(\varkappa)\right| &\leq \mathbb{L}\left\|\mathcal{N}_1'\right\|_*\|\mathcal{A}_1x\|_0+\mathbb{L}\|\mathcal{N}_1\|_*\left\|(\mathcal{A}_1x)'\right\|_0+\mathbb{L}\left\|\mathcal{Q}_1'\right\|_0\\
&+\left(2\mathbb{L}\left\|\mathcal{N}_1'\right\|_*\|r_1\|+2\mathbb{L}\|\mathcal{N}_1\|_*\delta_1+\delta_2\right)\eta_2\\
&+\left(2\mathbb{L}\|\mathcal{N}_1\|_*\|r_1\|+\delta_1\right)\gamma_2\left\|x'\right\|_0\gamma_1\\
&+\mathbb{L}\left(\left\|c_1'\right\|\sup_{\|u\|\leq D}|\Gamma(u)|+\|c_1\|\left\|x'\right\|_0\sup_{\|u\|\leq D}\left|\Gamma'(u)\right|\right)\\
&+\mathbb{L}\left(\left\|c_1'\right\|\|r_1\|+\|c_1\|\delta_1\right)\sup_{\|u\|\leq D}|\Gamma(u)|\\
&+\mathbb{L}\|\mathcal{Q}_1\|_0\|r_1\|\gamma_2\left\|x'\right\|_0\sup_{\|u\|\leq D}\left|\Gamma'(u)\right|\\
&=\mathbb{L}\left\|\mathcal{N}_1'\right\|_*D+\mathbb{L}\|\mathcal{N}_1\|_*D+\mathbb{L}\left\|\mathcal{Q}_1'\right\|_0\\
&+\left(2\mathbb{L}\left\|\mathcal{N}_1'\right\|_*\|r_1\|+2\mathbb{L}\|\mathcal{N}_1\|_*\delta_1+\delta_2\right)\eta_2\\
&+\left(2\mathbb{L}\|\mathcal{N}_1\|_*\|r_1\|+\delta_1\right)\gamma_2D\gamma_1\\
&+\mathbb{L}\left(\left\|c_1'\right\|\sup_{\|u\|\leq D}|\Gamma(u)|+\|c_1\|D\sup_{\|u\|\leq D}\left|\Gamma'(u)\right|\right)\\
&+\mathbb{L}\left(\left\|c_1'\right\|\|r_1\|+\|c_1\|\delta_1\right)\sup_{\|u\|\leq D}|\Gamma(u)|\\
&+\mathbb{L}\|\mathcal{Q}_1\|_0\|r_1\|\gamma_2D\sup_{\|u\|\leq D}\left|\Gamma'(u)\right|\\
&=D_1.
\end{aligned}$$

Thus, one can find a constant $D_1>0$ in a way that

$$\sup_{\varkappa\in[0,\ell]}\left|(\mathcal{A}_1x)''(\varkappa)\right|\leq D_1 \text{ and } \left\{(\mathcal{A}_1x)':\ x\in\mathbb{M}\right\}\subset\mathcal{C}_\ell$$

By the Ascoli Arzela Lemma, the sequence $\{x_{n_k}\}$ has a sub-sequence, denoted again for simplicity as $\{x_{n_k}\}$, such that $(\mathcal{A}_1x_{n_k})'\to z_0\in\mathcal{C}_\ell$. Since the differentiation operator $\dfrac{d}{d\varkappa}$ is closed, it follows that $z_0=(x_0)'$. Thus, we conclude that $x_0\in\mathcal{C}_\ell^1$ and that the sequence $\{\mathcal{A}_1x_n\}$ is contain within a compact set. Consequently, $\mathcal{A}_1$ is a compact operator. Consider that

$\{x_n\} \subset \mathbb{M}$, $x \in \mathbb{S}$, $x_n \to x$, then $\|x_n - x\|_0 \to 0$ and $\left\|x_n' - x'\right\|_0 \to 0$ as $n \to +\infty$,then one have

$$\begin{aligned}\|\mathcal{A}_1 x_n - \mathcal{A}_1 x\|_0 = \sup_{\varkappa\in[0,\ell]} &\Big| \Upsilon(\varkappa)\left(\Upsilon^{-1}(\ell) - I\right)^{-1} \int_{\varkappa}^{\varkappa+\ell} \Upsilon^{-1}(s)\left[\left(2\mathbb{L}\mathcal{N}_1(s)r_1(s) - r_1'(s)\right)\right. \\ &\times (\pounds(x_n(s-\wp(s))) - \pounds(x(s-\wp(s)))) - \mathbb{L}c_1(s)(\Gamma(x_n(s)) - \Gamma(x(s))) \\ &+\mathbb{L}c_1(s)r_1(s)(\Gamma(x_n(s-\wp(s))) - \Gamma(x(s-\wp(s))))]\,ds| \\ &\leq \mu\ell\left[(2\mathbb{L}\|\mathcal{N}_1\|_* \|r_1\| + \delta_1)\, l\, |x_n(\varkappa) - x(\varkappa)|\right. \\ &\left.+\mathbb{L}\|c_1\|(1+\|r_1\|) \sup_{\varkappa\in[0,\ell]} |\Gamma(x_n(\varkappa)) - \Gamma(x(\varkappa))|\right],\end{aligned}$$

and

$$\begin{aligned}\left\|(\mathcal{A}_1 x_n)' - (\mathcal{A}_1 x)'\right\|_0 = \sup_{\varkappa\in[0,\ell]} &\Big| \mathbb{L}\mathcal{N}_1(\varkappa)\left((\mathcal{A}_1 x_n)(\varkappa) - (\mathcal{A}_1 x)(\varkappa)\right) \\ &+ \left(2\mathbb{L}\mathcal{N}_1(\varkappa) r_1(\varkappa) - r_1'(\varkappa)\right) \\ &\times (\pounds(x_n(\varkappa - \wp(\varkappa))) - \pounds(x(\varkappa - \wp(\varkappa)))) \\ &- \mathbb{L}c_1(\varkappa)\left(\Gamma(x_n(\varkappa)) - \Gamma(x(\varkappa))\right) \\ &+ \mathbb{L}c_1(\varkappa) r_1(\varkappa)\left(\Gamma(x_n(\varkappa - \wp(\varkappa))) - \Gamma(x(\varkappa - \wp(\varkappa)))\right)\Big| \\ \leq\ &\mathbb{L}\|\mathcal{N}_1\|_* \|\mathcal{A}_1 x_n - \mathcal{A}_1 x\|_0 \\ &+ (2\mathbb{L}\|\mathcal{N}_1\|_* \|r_1\| + \delta_1)\, l\, |x_n(\varkappa) - x(\varkappa)| \\ &+ \mathbb{L}\|c_1\|(1+\|r_1\|) \sup_{\varkappa\in[0,\ell]} |\Gamma(x_n(\varkappa)) - \Gamma(x(\varkappa))| \\ =\ &(1 + \mathbb{L}\|\mathcal{N}_1\|_* \mu\ell)\left[(2\mathbb{L}\|\mathcal{N}_1\|_* \|r_1\| + \delta_1)\, l\, |x_n(\varkappa) - x(\varkappa)|\right. \\ &\left.+ \mathbb{L}\|c_1\|(1+\|r_1\|) \sup_{\varkappa\in[0,\ell]} |\Gamma(x_n(\varkappa)) - \Gamma(x(\varkappa))|\right].\end{aligned}$$

When $\|x_n - x\|_1 \to 0$ as $n \to +\infty$, $|x_n(\varkappa) - x(\varkappa)|$ for $\varkappa \in [0, \ell]$ uniformly. From the continuity of Γ, we have $\|\mathcal{A}_1 x_n - \mathcal{A}_1 x\|_0 \to 0$, $\left\|(\mathcal{A}_1 x_n)' - (\mathcal{A}_1 x)'\right\|_0 \to 0$, as a result of this, $\mathcal{A}_1$ is continuous

For all $x, y \in \mathbb{M}$, we have

$$\begin{aligned}\|\mathcal{B}_1 x - \mathcal{B}_1 y\|_1 &= \|\mathcal{B}_1 x - \mathcal{B}_1 y\|_0 + \left\|(\mathcal{B}_1 x)' - (\mathcal{B}_1 y)'\right\|_0 \\ &\leq l\left((1+\gamma_2 D)\|r_1\| + \delta_1\right)\|x - y\|_0 \\ &\leq \rho\|x - y\|_0 \\ &\leq \rho\|x - y\|_1,\end{aligned}$$

therefore $\mathcal{B}_1$ is a contraction operator. Thus, the conditions of Theorem 3.1 are satisfied and there is a $w \in \mathbb{M}$ such that $w = \mathcal{A}_1 w + \mathcal{B}_1 w$. It is a ℓ-periodic solution for (3.3). Since $v(\varkappa) = u(L\varkappa)$, $\mathcal{Q}_1(\varkappa) = \mathcal{Q}(\mathbb{L}\varkappa)$, $\mathcal{N}_1(\varkappa) = \mathcal{N}(\mathbb{L}\varkappa)$, $r_1(\varkappa) = r(\mathbb{L}\varkappa)$ and $c_1(\varkappa) = c(\mathbb{L}\varkappa)$, then (3.2) has a T -periodic solution. □

Remark 3.1. *It is readily apparent that relation (3.2) reduces to form (3.1) in the special case when the parameters are constant, i.e., when* $r(\varkappa) = r$, $\varsigma(\varkappa) = \varsigma$, $c(\varkappa) = c$ *and* $\pounds(u(\varkappa - r(\varkappa))) = u(\varkappa - \varsigma)$, *yielding specific results concerning the existence of periodic solutions to equation (3.1). This becomes clearly evident through the following theorem*

For a sufficiently small positive $\mathbb{L}$, (3.1) can be transformed as

$$\begin{aligned} &\frac{d}{d\varkappa}v(\varkappa) - r\frac{d}{d\varkappa}v(\varkappa-\tau) \\ &= \mathbb{L}\mathcal{Q}_1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)rv(\varkappa-\tau) - \mathbb{L}c\Gamma(v(\varkappa)) + \mathbb{L}cr\Gamma(v(\varkappa-\tau)), \end{aligned} \tag{3.23}$$

where $v(\varkappa) = u(\mathbb{L}\varkappa)$, $\ell = \dfrac{T}{\mathbb{L}}$, $\mathcal{Q}_1(\varkappa) = \mathcal{Q}(\mathrm{L}\varkappa), and \mathrm{N}_1(\varkappa) = \mathcal{N}(\mathbb{L}\varkappa)$.

Let

$$\mathbb{S} = \left\{\phi \in C^1(\mathbb{R}, \mathbb{R}^n),\ \|\phi\|_1 = \|\phi\|_0 + \|\phi'\|_0 < +\infty\right\},$$

and

$$\mathbb{M} = \left\{\phi \in C^1_\ell,\ \|\phi\|_1 \le H\right\},$$

then $\mathbb{M}$ is a bounded closed convex set of the Banach space $\mathbb{S}$.

Theorem 3.3 ([20])**.** *Suppose that* $\Gamma \in C^1(\mathbb{R}^n)$ *and* $\mathcal{Q}_1, \mathcal{N}_1 \in C^1_\ell$. *If there exists a constant* $H > 0$ *such that*

$$\frac{\sup\limits_{|u|\le H} |\Gamma(u)|}{H} < \frac{1}{(1 + (1 + \mathbb{L}\|\mathcal{N}_1\|_*)\mu\ell)\,\mathbb{L}c},$$

and that

$$|r| < \frac{1 - (1 + (1 + \mathbb{L}\|\mathcal{N}_1\|_*)\mu\ell)\,\mathbb{L}c\frac{\sup\limits_{|u|\le H}|\Gamma(u)|}{H}}{1 + 2\|\mathcal{N}_1\|_*(1 + (1 + \mathbb{L}\|\mathcal{N}_1\|_*)\mu\ell)\,\mathbb{L} + (1 + (1 + \mathbb{L}\|\mathcal{N}_1\|_*)\mu\ell)\,\mathbb{L}c\frac{\sup\limits_{|u|\le H}|\Gamma(u)|}{H}},$$

and

$$\|\mathcal{Q}_1\|_0 \le \frac{(1 - |r|)H}{(1 + (1 + \mathbb{L}\|\mathcal{N}_1\|_*)\mu\ell)\,\mathbb{L}} - 2\|\mathcal{N}_1\|_*|r|H - c(1 + |r|)\sup_{|u|\le H}|\Gamma(u)|,$$

where $\|\mathcal{N}_1\|_* = \sup\limits_{t\in[0,\ell]} |\mathcal{N}_1(\varkappa)|_*$ *and*

$$\mu = \sup_{\varkappa\in[0,\ell]}\left(\sup_{\varkappa\le s\le\varkappa+\ell}\left|[\mathrm{Y}(s)\left(\mathrm{Y}^{-1}(\ell) - I\right)\mathrm{Y}^{-1}(\varkappa)]^{-1}\right|\right).$$

Then (3.1) has a T*-periodic solution*

3.2.2 Examples

Example 3.1. *Consider the following neutral differential system*

$$\begin{aligned} &\frac{d}{d\varkappa}u(\varkappa) - r\frac{d}{d\varkappa}u(\varkappa-\varsigma) \\ = \;& \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa) + \mathcal{N}(\varkappa)ru(\varkappa-\varsigma) - c\Gamma(u(\varkappa)) + cr\Gamma(u(\varkappa-\varsigma)), \end{aligned} \tag{3.24}$$

where $T = 2\pi$, $c = 1$, $r = \frac{1}{80}$, $\varsigma = 2$,

$$\mathcal{N}(\varkappa) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \mathcal{Q}(\varkappa) = \begin{pmatrix} 0 \\ 0.01\cos(\varkappa) \end{pmatrix}, \quad \Gamma(u(\varkappa)) = \begin{pmatrix} 0 \\ \sin(u(\varkappa)) \end{pmatrix}.$$

For $\mathbb{L} = 0.25$, *(3.24) can be transformed as*

$$\begin{aligned} &\frac{d}{d\varkappa}v(\varkappa) - r\frac{d}{d\varkappa}v(\varkappa-\tau) \\ &= \mathbb{L}\mathcal{Q}1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)rv(\varkappa-\tau) - \mathbb{L}c\Gamma(v(\varkappa)) + \mathbb{L}cr\Gamma(v(\varkappa-\tau)), \end{aligned} \tag{3.25}$$

where $v(\varkappa) = u(0.25\varkappa)$, $\ell = 8\pi$, $\tau = 8$,

$$\mathcal{Q}_1(\varkappa) = \begin{pmatrix} 0 \\ 0.01\cos(0.25\varkappa) \end{pmatrix}, \quad \mathcal{N}_1(\varkappa) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Since the matrix $\mathcal{N}_1$ *has eigenvalues with non-zero real parts, the system*

$$\frac{d}{d\varkappa}v(\varkappa) = \mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa)$$

is noncritical. Let $H = 30$, *then all conditions of Theorem 3.3 are satisfied and hence (3.24) has a* 2π*-periodic solution.*

Example 3.2. *Consider the following neutral differential system*

$$\begin{aligned} &\frac{d}{dt}u(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\pounds(u(\varkappa-\varsigma(\varkappa))) = \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa) \\ &+\mathcal{N}(\varkappa)r(\varkappa)\pounds(u(\varkappa-\varsigma(\varkappa))) - c(\varkappa)\Gamma(u(\varkappa)) + c(\varkappa)r(\varkappa)\Gamma(u(\varkappa-\varsigma(\varkappa))), \end{aligned} \tag{3.26}$$

where $T = 2\pi$, $\varsigma(\varkappa) = 2\sin(\varkappa)$, $r(\varkappa) = \frac{1}{80}\sin(\varkappa)$, $c(\varkappa) = \sin(\varkappa)$,

$$\mathcal{Q}(\varkappa) = \begin{pmatrix} 0.01\cos(\varkappa) \\ 0.05\sin(\varkappa) \\ 0.01\cos(\varkappa) \end{pmatrix}, \mathcal{N}(\varkappa) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} & 0 \\ -3 & 0 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 3 \end{pmatrix}, \Gamma(u(\varkappa)) = \begin{pmatrix} 0 \\ 0 \\ \sin(u(\varkappa)) \end{pmatrix},$$

and $\pounds(u(\varkappa)) = \begin{pmatrix} 0 \\ 0 \\ 0.5 + 0.5\arctan(u(\varkappa)) \end{pmatrix}$. *For a sufficiently small positive* $\mathbb{L} = 0.25$, *(3.26) can be transformed as*

$$\begin{aligned} &\frac{d}{d\varkappa}v(\varkappa) - r_1(\varkappa)\frac{d}{d\varkappa}\pounds(v(\varkappa - \wp(\varkappa)) \\ &= \mathbb{L}\mathcal{Q}_1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)r_1(\varkappa)\pounds(v(\varkappa - \wp(\varkappa)) \\ &- \mathbb{L}c_1(\varkappa)\Gamma(v(\varkappa)) + \mathbb{L}c_1(\varkappa)r_1(\varkappa)\Gamma(v(\varkappa - \wp(\varkappa))), \end{aligned} \tag{3.27}$$

where $v(\varkappa) = u(0.25\varkappa)$, $\ell = \dfrac{T}{\mathbb{L}} = 8\pi$, $\wp(\varkappa) = \dfrac{\varsigma(0.25\varkappa)}{\mathbb{L}} = 8\sin(0.25\varkappa)$,

$r_1(\varkappa) = \dfrac{1}{320}\sin(0.25\varkappa)$, $c_1(\varkappa) = \sin(0.25\varkappa)$, $\mathcal{Q}_1(\varkappa) = \begin{pmatrix} 0.01\cos(0.25\varkappa) \\ 0.05\sin(0.25\varkappa) \\ 0.01\cos(0.25\varkappa) \end{pmatrix}$,

$\mathcal{N}_1(\varkappa) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} & 0 \\ -3 & 0 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 3 \end{pmatrix}$. *We will calculate* μ *with*

$$\begin{aligned} \mu &= \sup_{\varkappa\in[0,\ell]}\left(\sup_{\varkappa\le s\le\varkappa+\ell}\left|\left[\Upsilon(s)\left(\Upsilon^{-1}(\ell) - I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}\right|_*\right) \\ &= \sup_{\varkappa\in[0,8\pi]}\left(\sup_{\varkappa\le s\le\varkappa+8\pi}\left|\left[\Upsilon(s)\left(\Upsilon^{-1}(8\pi) - I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}\right|_*\right), \end{aligned}$$

we have $\Upsilon(\varkappa) = e^{\int_0^{\varkappa}\mathcal{N}_1(s)ds} = \exp(\varkappa\mathcal{N}_1)$, *where*

$$\mathcal{N}_1(\varkappa) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} & 0 \\ -3 & 0 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 3 \end{pmatrix} = PBP^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and therefore

$$\begin{aligned} \Upsilon(\varkappa) &= e^{\int_0^{\varkappa}\mathcal{N}_1(s)ds} = \exp(\mathcal{N}_1) = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} e^{\frac{3}{2}\varkappa} & 0 & 0 \\ 0 & e^{3\varkappa} & 0 \\ 0 & 0 & e^{3\varkappa} \end{pmatrix}\begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{3\varkappa} - e^{\frac{3}{2}\varkappa} & e^{3\varkappa} - e^{\frac{3}{2}\varkappa} & 0 \\ 2e^{\frac{3}{2}\varkappa} - 2e^{3\varkappa} & 2e^{\frac{3}{2}\varkappa} - e^{3\varkappa} & 0 \\ e^{\frac{3}{2}\varkappa} - e^{3\varkappa} & e^{\frac{3}{2}\varkappa} - e^{3\varkappa} & e^{3\varkappa} \end{pmatrix}, \end{aligned}$$

and

$$\Upsilon^{-1}(\varkappa)=\begin{pmatrix} -\left(e^{-\frac{3}{2}\varkappa}-2e^{-3\varkappa}\right) & -\left(e^{-\frac{3}{2}\varkappa}-e^{-3\varkappa}\varkappa\right) & 0 \\ \left(2e^{-\frac{3}{2}\varkappa}-2e^{-3\varkappa}\right) & \left(2e^{-\frac{3}{2}\varkappa}-e^{-3\varkappa}\right) & 0 \\ \left(e^{-\frac{3}{2}\varkappa}-e^{-3\varkappa}\right) & \left(e^{-\frac{3}{2}\varkappa}-e^{-3\varkappa}\right) & e^{-3\varkappa} \end{pmatrix},$$

for $\varkappa=\ell=8\pi$,*we obtain*

$$\Upsilon^{-1}(\ell)=\Upsilon^{-1}(8\pi)=\begin{pmatrix} -4.2412\times10^{-17} & -4.2412\times10^{-17} & 0 \\ 8.4823\times10^{-17} & 8.4823\times10^{-17} & 0 \\ 4.2412\times10^{-17} & 4.2412\times10^{-17} & 1.4246\times10^{-52} \end{pmatrix}$$
$$=\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

we find

$$\left[\Upsilon(s)\left(\Upsilon^{-1}(8\pi)-I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}$$
$$=\begin{pmatrix} -1&-1&0\\2&1&0\\1&1&-1\end{pmatrix}\begin{pmatrix} e^{\frac{3}{2}(\varkappa-s)}&0&0\\0&e^{3(\varkappa-s)}&0\\0&0&e^{3(\varkappa-s)}\end{pmatrix}\begin{pmatrix} -1&-1&0\\2&1&0\\1&0&1\end{pmatrix}.$$

Thus

$$\sup_{\varkappa\le s\le\varkappa+8\pi}\left|\left[\Upsilon(s)\left(\Upsilon^{-1}(8\pi)-I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}\right|_{*}=9,$$

then

$$\mu=\sup_{\varkappa\in[0,8\pi]}\left(\sup_{\varkappa\le s\le\varkappa+8\pi}\left|\left[\Upsilon(s)\left(\Upsilon^{-1}(8\pi)-I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}\right|_{*}\right)=9.$$

Knowing that $\|\mathcal{N}_1\|_*=6$, $\|r_1\|=\frac{1}{80}$, $\delta_1=\left\|r_1'\right\|=\frac{1}{320}$, $\gamma_2=\left\|1-\wp'(\varkappa)\right\|=3$, $|\pounds(0)|=0.5$, $l=0.5$ *and therefore* $\eta_1=566.5$. *The matrix* $\mathcal{N}_1(\varkappa)$ *have eigenvalue with non-zero real part, the systems* $\frac{d}{d\varkappa}v(\varkappa)=\mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa)$ *is noncritical. Putting* $D=8$ *and* $\rho=0.16$, *then all conditions of the above Theorem 3.2 are met; hence, the systems (3.26) has a* 2π*-periodic solutions.*

3.3 Asymptotic stability of periodic solutions

In this section of the chaptre, we will discuss the stability of the periodic solution. Assuming that condition of the above theorem 3.2 fulfill, then a T-periodic solution u^* exists, representing an equilibrium for (3.2). Defining $v(\varkappa) = u(\varkappa) - u^*(\varkappa)$, we transform (1.4) into

$$\begin{aligned}
&\frac{d}{d\varkappa}v(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\left(\pounds\left(v\left(\varkappa - \wp(\varkappa)\right) + u^*\left(\varkappa - \wp(\varkappa)\right)\right) - \pounds(u^*(\varkappa - \wp(\varkappa)))\right)\\
&= \mathcal{N}(\varkappa)v(\varkappa) + \mathcal{N}(\varkappa)r(\varkappa)\left(\pounds\left(v\left(\varkappa - \wp(\varkappa)\right) + u^*\left(\varkappa - \wp(\varkappa)\right)\right) - \pounds(u^*(\varkappa - \wp(\varkappa)))\right)\\
&- c(\varkappa)\left(\Gamma\left(v\left(\varkappa\right) + u^*\left(\varkappa\right)\right) - \Gamma(u^*(\varkappa))\right)\\
&+ c(\varkappa)r(\varkappa)\left(\Gamma\left(v\left(\varkappa - \wp(\varkappa)\right) + u^*\left(\varkappa - \wp(\varkappa)\right)\right) - \Gamma(u^*(\varkappa - \wp(\varkappa)))\right). \qquad (3.28)
\end{aligned}$$

It is evident that equation (3.28) admits the trivial solution. We now apply Krasnoselskii's fixed point theorem to establish the asymptotic stability of this trivial solution. Let $\mathbb{S}$ denote the Banach space of bounded continuous functions $\vartheta : [m(0), \infty) \to \mathbb{R}^n$ equipped with the supremum norm $\|\cdot\|$, where $m(0) = \inf\{\varkappa - \wp(\varkappa) \mid, \varkappa \geq 0\}$. Additionally, for an initial function Φ, we define its norm as $\|\Phi\| = \sup_{\varkappa\in[m(0),0]} |\Phi(\varkappa)|$, which, despite using the same notation, should not be confused with the norm in $\mathbb{S}$.

Proposition 3.1 ([18]). *Assume the system*

$$y'(\varkappa) = \mathcal{N}(\varkappa)y(\varkappa), \qquad (3.29)$$

where $\varkappa \mapsto \Theta(\varkappa)$ is a fundamental matrix solution define on an open interval K. The state transition matrix is given by $\Psi(\varkappa, \varsigma) = \Theta(\varkappa)\Theta^{-1}(\varsigma)$. The state transition matrix holds the Chapman-Kolmogorov identity:

$$\Psi(\varsigma, \varsigma) = I, \quad \Psi(\varkappa, s)\Psi(s, \varsigma) = \Psi(\varkappa, \varsigma).$$

Furthermore, it satisfies the following properties:

$$\Psi(\varkappa, s)^{-1} = \Psi(s, \varkappa), \quad \frac{\partial}{\partial s}\Psi(\varkappa, s) = -\Psi(\varkappa, s)\mathcal{N}(s).$$

Definition 3.2. *(1). The trivial solution of (3.28) is considered stable if, for every $0 < \xi$, $\exists$ a $0 < \delta$ such that whenever an initial function $\Phi \in \mathcal{C}([m(0), 0], \mathbb{R}^n)$ with $\|\Phi\| < \delta$ is given, the corresponding solution $x(\varkappa, 0, \Phi)$ of (3.28) holds $|x(\varkappa, 0, \Phi)| < \xi$ for all $\varkappa \geq 0$.*
(2). The trivial solution of (3.28) is asymptotically stable if it is stable and $\exists$ a constant $\eta > 0$ such that for any initial function $\Phi : [m(0), 0] \to \mathbb{R}^n$, $\Phi \in \mathcal{C}([m(0), 0], \mathbb{R}^n)$, with $\|\Phi\| < \eta$, the corresponding solutions $x(\varkappa, 0, \Phi)$ of (3.28) holds $\lim_{\varkappa\to\infty} x(\varkappa, 0, \Phi) = 0$.

Theorem 3.4. *If all the conditions of the above theorem 3.2 are holds and* Γ *fulfills the locally Lipschitz condition with a Lipschitz constant* R*, then further consider that*

$$\Theta(\varkappa) \to 0 \quad as \quad \varkappa \to \infty, \quad \varkappa - \wp(\varkappa) \to \infty, \tag{3.30}$$

and

$$l\|r\| < 1. \tag{3.31}$$

Moreover, suppose $\exists$ *a constant* $\daleth > D$ *in a manner that*

$$\lambda l \delta_1 < 1, \tag{3.32}$$

$$(1 - \lambda l \delta_1) \daleth > 2\lambda l \delta_1 D, \tag{3.33}$$

$$\sup_{\|u\| \le D + \daleth} |\Gamma(u)| + \sup_{\|u\| \le D} |\Gamma(u)| < \frac{(1 - \lambda l \delta_1) \daleth - 2\lambda l \delta_1 D}{\lambda \|c\|}, \tag{3.34}$$

$$\|r\| < \frac{(1 - \lambda l \delta_1) \daleth - 2\lambda l \delta_1 D - \lambda \|c\| \left(\sup_{\|u\| \le D + \daleth} |\Gamma(u)| + \sup_{\|u\| \le D} |\Gamma(u)|\right)}{l\daleth + 2\lambda l \|\mathcal{N}\|_* (2D + \daleth) + \lambda \|c\| \left(\sup_{\|u\| \le D + \daleth} |\Gamma(u)| + \sup_{\|u\| \le D} |\Gamma(u)|\right)}, \tag{3.35}$$

and

$$\|\Phi\| < \left[\frac{(1 - l\|r\|)\daleth - \lambda l (2\|\mathcal{N}\|_* \|r\| + \delta_1)(2D + \daleth)}{\theta(1 + l\|r\|)} - \frac{r\|c\|(1 + \|r\|)\left(\sup_{\|v\| \le D + \daleth} |\Gamma(v)| + \sup_{\|v\| \le D} |\Gamma(v)|\right)}{\theta(1 + l\|r\|)}\right], \tag{3.36}$$

where

$$\theta = \sup_{\varkappa \ge 0} |\Psi(\varkappa, 0)|_* \quad and \quad \lambda = \sup_{\varkappa \ge 0} \left| \int_0^{\varkappa} \Psi(\varkappa, s)\, ds \right|_*. \tag{3.37}$$

Therefore, the trivial solution of (3.28) converges toward the point zero as time approaches infinity $\varkappa \to \infty$.

Proof. Based on the conditions expressed in (3.34), (3.35) and (3.36), the subsequent inequality holds:

$$\begin{aligned}
&\theta\left(1+l\left\|r\right\|\right)\left\|\Phi\right\|+l\left\|r\right\|\daleth+\lambda l\left(2\left\|\mathcal{N}\right\|_{*}\left\|r\right\|+\delta_{1}\right)\left(2D+\daleth\right)\\
&\quad+\lambda\left\|c\right\|\left(1+\left\|r\right\|\right)\left(\sup_{\left\|v\right\|\leq D+\daleth}\left|\Gamma\left(v\right)\right|+\sup_{\left\|v\right\|\leq D}\left|\Gamma\left(v\right)\right|\right)\\
&\leq\daleth-l\left\|r\right\|\daleth-\lambda l\left(2\left\|\mathcal{N}\right\|_{*}\left\|r\right\|+\delta_{1}\right)\left(2D+\daleth\right)\\
&\quad-\lambda\left\|c\right\|\left(1+\left\|r\right\|\right)\left(\sup_{\left\|v\right\|\leq D+\daleth}\left|\Gamma\left(v\right)\right|+\sup_{\left\|v\right\|\leq D}\left|\Gamma\left(v\right)\right|\right)\\
&l\left\|r\right\|\daleth+\lambda l\left(2\left\|\mathcal{N}\right\|_{*}\left\|r\right\|+\delta_{1}\right)\left(2D+\daleth\right)\\
&\quad+\lambda\left\|c\right\|\left(1+\left\|r\right\|\right)\left(\sup_{\left\|v\right\|\leq D+\daleth}\left|\Gamma\left(v\right)\right|+\sup_{\left\|v\right\|\leq D}\left|\Gamma\left(v\right)\right|\right)\\
&=\daleth.
\end{aligned}\tag{3.38}$$

For a given initial function Φ, $\exists$ a unique solution v to the system (3.28). Assume

$$\aleph_{\Phi}=\left\{\vartheta\in\mathbb{S},\ \left\|\vartheta\right\|\leq\daleth,\ \vartheta\left(\varkappa\right)=\Phi\left(\varkappa\right),\ \text{if}\ \varkappa\in\left[m\left(0\right),0\right],\ \vartheta(\varkappa)\to0\ \text{as}\ \varkappa\to\infty\right\},$$

Here, $\aleph_{\Phi}$ represents a closed convex bounded subset within the space $\mathbb{S}$. Suppose v is any solution to (3.28), express (3.28) in the given way:

$$\begin{aligned}
&\frac{d}{d\varkappa}\Big[v(\varkappa)-r(\varkappa)\Big(\pounds\Big(v(\varkappa-\wp(\varkappa))+u^{*}(\varkappa-\wp(\varkappa))\Big)-\pounds(u^{*}(\varkappa-\wp(\varkappa)))\Big)\Big]\\
&=\mathcal{N}(\varkappa)v(\varkappa)+\Big(\mathcal{N}(\varkappa)r(\varkappa)-r'(\varkappa)\Big)\Big(\pounds\Big(v(\varkappa-\wp(\varkappa))+u^{*}(\varkappa-\wp(\varkappa))\Big)-\pounds(u^{*}(\varkappa-\wp(\varkappa)))\Big)\\
&\quad-c(\varkappa)\Big(\Gamma\Big(v(\varkappa)+u^{*}(\varkappa)\Big)-\Gamma(u^{*}(\varkappa))\Big)\\
&\quad+c(\varkappa)r(\varkappa)\Big(\Gamma\Big(v(\varkappa-\wp(\varkappa))+u^{*}(\varkappa-\wp(\varkappa))\Big)-\Gamma(u^{*}(\varkappa-\wp(\varkappa)))\Big).
\end{aligned}$$

Given that Θ is a fundamental matrix solutions of (3.29), we obtain:

$$\begin{aligned}
&\frac{d}{d\varkappa}\left\{\Theta^{-1}\left(\varkappa\right)\left[v(\varkappa)-r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^{*}\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^{*}(\varkappa-\wp(\varkappa)))\right)\right]\right\}\\
&=\left\{\frac{d}{d\varkappa}\Theta^{-1}\left(\varkappa\right)\right\}\left[v(\varkappa)-r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^{*}\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^{*}(\varkappa-\wp(\varkappa)))\right)\right]\\
&\quad+\Theta^{-1}\left(\varkappa\right)\frac{d}{d\varkappa}\left[v(\varkappa)-r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^{*}\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^{*}(\varkappa-\wp(\varkappa)))\right)\right].
\end{aligned}$$

Based on proposition 3.1, we have

$$\frac{d}{d\varkappa}\Theta^{-1}\left(\varkappa\right)=-\Theta^{-1}\left(\varkappa\right)\mathcal{N}\left(\varkappa\right).$$

Thus,

$$\frac{d}{d\varkappa}\left\{\Theta^{-1}(\varkappa)\left[v(\varkappa)-r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)\right]\right\}$$
$$=-\Theta^{-1}(\varkappa)\mathcal{N}(\varkappa)\left[v(\varkappa)-r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)\right]$$
$$+\Theta^{-1}(\varkappa)\left\{\mathcal{N}(\varkappa)v(\varkappa)+\Big(\mathcal{N}(\varkappa)r(\varkappa)-r'(\varkappa)\Big)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)\right.\right.$$
$$\left.-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)-c(\varkappa)\left(\Gamma\left(v(\varkappa)+u^*(\varkappa)\right)-\Gamma(u^*(\varkappa))\right)+c(\varkappa)r(\varkappa)\}$$
$$\times\left(\Gamma\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\Gamma(u^*(\varkappa-\wp(\varkappa))\right)$$
$$=\Theta^{-1}(\varkappa)\left\{\Big(2\mathcal{N}(\varkappa)r(\varkappa)-r'(\varkappa)\Big)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)\right.$$
$$-c(\varkappa)\left(\Gamma\left(v(\varkappa)+u^*(\varkappa)\right)-\Gamma(u^*(\varkappa))\right)+c(\varkappa)r(\varkappa)$$
$$\left.\left(\Gamma\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\Gamma(u^*(\varkappa-\wp(\varkappa)))\right)\right\}$$

When both sides of the equation above are integrated from 0 to $\varkappa$, the result is

$$\begin{aligned}
&\Theta^{-1}(\varkappa)\left[v(\varkappa)-r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)\right]\\
&-\Theta^{-1}(0)\left[v(0)-r(0)\left(\pounds\left(v\left(-\wp(0)\right)+u^*\left(-\wp(0)\right)\right)-\pounds(u^*(-\wp(0)))\right)\right]\\
=&\int_0^{\varkappa-1}\Theta^{-1}(s)\left\{\Big(2\mathcal{N}(s)r(s)-r'(s)\Big)\left(\pounds\left(v\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)-\pounds(u^*(s-\wp(s)))\right)\right.\\
&-c(s)\left(\Gamma\left(v(s)+u^*(s)\right)-\Gamma(u^*(s))\right)+c(s)r(s)(\Gamma((v\left(s-\wp(s)\right)\\
&\left.+u^*\left(s-\wp(s)\right)-\Gamma(u^*(s-\wp(s)))\right\}ds,
\end{aligned}\tag{3.39}$$

Multiplying the expression above by the matrix $\Theta(\varkappa)$ and taking into account Proposition 3.1 with $\Theta(\varkappa)\Theta^{-1}(\varkappa)=I$, $\Theta(\varkappa)\Theta^{-1}(s)=\Psi(\varkappa,s)$ and $\Theta(\varkappa)\Theta^{-1}(0)=\Psi(\varkappa,0)$, we get

$$\begin{aligned}
v(\varkappa)=&\,\Psi(\varkappa,0)\Big[\Phi(0)-r(0)\Big(\pounds\Big(v(-\wp(0))+u^*(-\wp(0))\Big)-\pounds\Big(u^*(-\wp(0))\Big)\Big)\Big]\\
&+r(\varkappa)\Big(\pounds\Big(v(\varkappa-\wp(\varkappa))+u^*(\varkappa-\wp(\varkappa))\Big)-\pounds\Big(u^*(\varkappa-\wp(\varkappa))\Big)\Big)\\
&+\int_0^{\varkappa}\Psi(\varkappa,s)\Big\{(2\mathcal{N}(s)r(s)-r'(s))\Big(\pounds\Big(v(s-\wp(s))+u^*(s-\wp(s))\Big)\\
&\quad-\pounds\Big(u^*(s-\wp(s))\Big)\Big)-c(s)\Big(\Gamma\Big(v(s)+u^*(s)\Big)-\Gamma\Big(u^*(s)\Big)\Big)\\
&\quad+c(s)r(s)\Big(\Gamma\Big(v(s-\wp(s))+u^*(s-\wp(s))\Big)-\Gamma\Big(u^*(s-\wp(s))\Big)\Big)\Big\}ds.
\end{aligned}$$

Now, considering $v(\varkappa)=\Phi(\varkappa)$ when $\varkappa\in[m(0),0]$ and for $\varkappa\geq 0$, it can be expressed as follows:

$$\begin{aligned}
v(\varkappa) &= \Psi(\varkappa,0)\Big[\Phi(0)-r(0)\Big(\pounds\Big(\Phi(-\wp(0))+u^*(-\wp(0))\Big)-\pounds\Big(u^*(-\wp(0))\Big)\Big)\Big]\\
&\quad+r(\varkappa)\Big(\pounds\Big(v(\varkappa-\wp(\varkappa))+u^*(\varkappa-\wp(\varkappa))\Big)-\pounds\Big(u^*(\varkappa-\wp(\varkappa))\Big)\Big)\\
&\quad+\int_0^{\varkappa}\Psi(\varkappa,s)\Big\{\Big(2\mathcal{N}(s)r(s)-r'(s)\Big)\Big(\pounds\Big(v(s-\wp(s))+u^*(s-\wp(s))\Big)\\
&\qquad-\pounds\Big(u^*(s-\wp(s))\Big)\Big)-c(s)\Big(\Gamma\Big(v(s)+u^*(s)\Big)-\Gamma\Big(u^*(s)\Big)\Big)\\
&\qquad+c(s)r(s)\Big(\Gamma\Big(v(s-\wp(s))+u^*(s-\wp(s))\Big)-\Gamma\Big(u^*(s-\wp(s))\Big)\Big)\Big\}ds.
\end{aligned}$$

With all $\vartheta\in\aleph_\Phi$, introduce $\mathcal{A}_2$ and $\mathcal{B}_2$ operators as:

$$(\mathcal{A}_2\vartheta)(\varkappa)=\begin{cases}0, \quad \varkappa\in[m(0),0],\\
\displaystyle\int_0^{\varkappa}\Psi(\varkappa,s)\Big\{(2\mathcal{N}(s)r(s)-r'(s))\Big(\pounds\Big(\vartheta(s-\wp(s))+u^*(s-\wp(s))\Big)\\
\quad-\pounds\Big(u^*(s-\wp(s))\Big)\Big)-c(s)\Big(\Gamma\Big(\vartheta(s)+u^*(s)\Big)\\
\quad-\Gamma\Big(u^*(s)\Big)\Big)+c(s)r(s)\Big(\Gamma\Big(\vartheta(s-\wp(s))\\
\quad+u^*(s-\wp(s))\Big)-\Gamma\Big(u^*(s-\wp(s))\Big)\Big)\Big\}ds, \quad \varkappa\geq 0\end{cases}$$

and

$$(\mathcal{B}_2\vartheta)(\varkappa)=\begin{cases}\Phi(\varkappa),\ \varkappa\in[m(0),0],\\
\Psi(\varkappa,0)\left[\Phi(0)-r(0)\left(\pounds\left(\Phi(-\wp(0))+u^*(-\wp(0))\right)-\pounds(u^*(-\wp(0)))\right)\right]\\
+r(\varkappa)\left(\pounds\left(\vartheta(\varkappa-\wp(\varkappa))+u^*(\varkappa-\wp(\varkappa))\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right),\ \varkappa\geq 0.\end{cases}$$

(i) Considering all $x,y\in\aleph_\Phi$, $x(\varkappa)\to 0$ and $x(\varkappa)\to 0$ as $\varkappa\to\infty$, then $\Psi(\varkappa,0)=\Theta(\varkappa)\Theta^{-1}(0)\to 0$ as $\varkappa\to 0$ hence,

$$\begin{aligned}
(\mathcal{B}_2y)(\varkappa) &= \Psi(\varkappa,0)\Big[\Phi(0)-r(0)\Big(\pounds\Big(\Phi(-\wp(0))+u^*(-\wp(0))\Big)\\
&\qquad-\pounds\Big(u^*(-\wp(0))\Big)\Big)\Big]\\
&\qquad+r(\varkappa)\Big(\pounds\Big(y(\varkappa-\wp(\varkappa))+u^*(\varkappa-\wp(\varkappa))\Big)\\
&\qquad-\pounds\Big(u^*(\varkappa-\wp(\varkappa))\Big)\Big)\\
&\to 0 \text{ as } \varkappa\to\infty,
\end{aligned}$$

and

$$\lim_{\varkappa \to +\infty} (\mathcal{A}_2 x)(\varkappa) = \lim_{\varkappa \to +\infty} \Bigg\{ \Theta(\varkappa) \int_0^{\varkappa} \Theta^{-1}(s) \Big\{ \Big(2\mathcal{N}(s) r(s) - r'(s) \Big)$$
$$\times \Big[\pounds\Big(x(s - \wp(s)) + u^*(s - \wp(s)) \Big) - \pounds\Big(u^*(s - \wp(s)) \Big) \Big]$$
$$- c(s) \Big(\Gamma\Big(x(s) + u^*(s) \Big) - \Gamma\Big(u^*(s) \Big) \Big)$$
$$+ c(s) r(s) \Big(\Gamma\Big(x(s - \wp(s)) + u^*(s - \wp(s)) \Big) - \Gamma\Big(u^*(s - \wp(s)) \Big) \Big) \Big\} ds \Bigg\}$$
$$= 0.$$

Then, as $\varkappa \to +\infty$, we have $(\mathcal{A}_2 x + \mathcal{B}_2 y)(\varkappa)$ converges to zero. Additionally:

$$\|\mathcal{A}_2 x\| = \sup_{\varkappa \geq m(0)} |(\mathcal{A}_2 x)(\varkappa)|$$
$$= \sup_{\varkappa \geq 0} \Bigg| \int_0^{\varkappa} \Psi(\varkappa, s) \Big\{ (2\mathcal{N}(s) r(s) - r'(s)) \Big(\pounds\Big(x(s - \wp(s)) + u^*(s - \wp(s)) \Big)$$
$$- \pounds\Big(u^*(s - \wp(s)) \Big) \Big) - c(s) \Big(\Gamma\Big(x(s) + u^*(s) \Big) - \Gamma\Big(u^*(s) \Big) \Big)$$
$$+ c(s) r(s) \Big(\Gamma\Big(x(s - \wp(s)) + u^*(s - \wp(s)) \Big) - \Gamma\Big(u^*(s - \wp(s)) \Big) \Big) \Big\} ds \Bigg|$$
$$\leq \Bigg\{ (2\|\mathcal{N}\|_* \|r\| + \|r'\|) \Big(\sup_{\|v\| \leq D + \daleth} |\pounds(v)| + \sup_{\|v\| \leq D} |\pounds(v)| \Big)$$
$$+ \|c\|(1 + \|r\|) \Big(\sup_{\|v\| \leq D + \daleth} |\Gamma(v)| + \sup_{\|v\| \leq D} |\Gamma(v)| \Big) \Bigg\}$$
$$\times \sup_{\varkappa \geq 0} \left| \int_0^{\varkappa} \Psi(\varkappa, s) ds \right|_*$$
$$\leq \Bigg[l(2D + \daleth)(2\|\mathcal{N}\|_* \|r\| + \delta_1)$$
$$+ \|c\|(1 + \|r\|) \Big(\sup_{\|v\| \leq D + \daleth} |\Gamma(v)| + \sup_{\|v\| \leq D} |\Gamma(v)| \Big) \Bigg],$$

and

$$
\begin{aligned}
\|\mathcal{B}_2 y\| &= \sup_{\varkappa \geq m(0)} |(\mathcal{B}_2 y)(\varkappa)| \\
&= \max\Bigg\{ \|\Phi\|, \sup_{\varkappa \geq 0} \Bigg| \Psi(\varkappa, 0) \Bigg[\Phi(0) - r(0) \Bigg(\pounds\left(\Phi(-\wp(0)) + u^*(-\wp(0))\right) \\
&\quad - \pounds(u^*(-\wp(0))) \Bigg) \Bigg] + r(\varkappa) \Bigg(\pounds\left(y(\varkappa - \wp(\varkappa)) + u^*(\varkappa - \wp(\varkappa))\right) \\
&\quad - \pounds(u^*(\varkappa - \wp(\varkappa))) \Bigg) \Bigg| \Bigg\} \\
&\leq \eta (1 + l \|r\|) \|\Phi\| + l \|r\| \beth.
\end{aligned}
$$

From above,

$$
\begin{aligned}
\|\mathcal{A}_2 x + \mathcal{B}_2 y\| &\leq \|\mathcal{A}_2 x\| + \|\mathcal{B}_2 y\| \\
&\leq r \Bigg[l (2D + \beth) (2 \|\mathcal{N}\|_* \|r\| + \delta_1) \\
&\quad + \|c\| (1 + \|r\|) \Bigg(\sup_{\|v\| \leq D + \beth} |\Gamma(v)| \\
&\quad + \sup_{\|v\| \leq \beth} |\Gamma(v)| \Bigg) \Bigg] \\
&\quad + \eta (1 + l \|r\|) \|\Phi\| + l \|r\| \beth.
\end{aligned}
$$

By the condition as expressed in (3.38), we arrive at $\|\mathcal{A}_2 x + \mathcal{B}_2 y\| \leq \beth$. This result ensures that $\mathcal{A}_2 x + \mathcal{B}_2 y$ is an element of $\aleph_\Phi$.

(ii) Given any $x \in \aleph_\Phi$, we have that $\|x\| \leq \beth$, and

$$
\left| (\mathcal{A}_2 x)' (\varkappa) \right| = 0, \; \varkappa \in [m(0), 0].
$$

Also for $\varkappa \geq 0$, using $\Theta'(\varkappa) = \mathcal{N}(\varkappa)\Theta(\varkappa)$, and with $\Psi(\varkappa,\varkappa) = I$, it implies the given equation

$$\begin{aligned}
(\mathcal{A}_2x)'(\varkappa) = \mathcal{N}(\varkappa)\int_0^{\varkappa}\Psi(\varkappa,s)\Big\{&\Big(2\mathcal{N}(s)r(s)-r'(s)\Big)\Big(\pounds\left(\vartheta\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)\\
&-\pounds\left(u^*(s-\wp(s))\right)\Big)-c(s)\Big(\Gamma\left(x\left(s\right)+u^*\left(s\right)\right)-\Gamma\left(u^*(s)\right)\Big)\\
&+c(s)r(s)\Big(\Gamma\left(x\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)-\Gamma\left(u^*(s-\wp(s))\right)\Big)\Big\}ds\\
&+\Big(2\mathcal{N}(\varkappa)r(\varkappa)-r'(\varkappa)\Big)\Big(\pounds\left(x\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)\\
&-\pounds\left(u^*(\varkappa-\wp(\varkappa))\right)\Big)\\
&-c(\varkappa)\Big(\Gamma\left(x\left(\varkappa\right)+u^*\left(\varkappa\right)\right)-\Gamma\left(u^*(\varkappa)\right)\Big)\\
&+c(\varkappa)r(\varkappa)\Big(\Gamma\left(x\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\Gamma\left(u^*(\varkappa-\wp(\varkappa))\right)\Big).
\end{aligned}$$

Thus, it implies the following

$$\begin{aligned}
\left|(\mathcal{A}_2x)'(\varkappa)\right| &\leq (1+\lambda\left\|\mathcal{N}\right\|_*)\Big[(2\left\|\mathcal{N}\right\|_*\left\|r\right\|+\delta_1)(2D+\daleth)\,l\\
&\quad+\left\|c\right\|(1+\left\|r\right\|)\Big(\sup_{\|v\|\leq D+\daleth}\left|\Gamma(v)\right|+\sup_{\|v\|\leq D}\left|\Gamma(v)\right|\Big)\Big]\\
&= \daleth_1,
\end{aligned}$$

It is essential to note that the derivative of $(\mathcal{A}_2x)'(\varkappa)$ at zero is interpreted as the left-hand derivative for $0 \geq \varkappa$ and as the right-hand derivative for $0 \leq \varkappa$. Consequently, it follows that $\left|(\mathcal{A}_2x)'(\varkappa)\right|$ remains bounded for all $x \in \aleph_\Phi$, implying that $\mathcal{A}_2\aleph_\Phi$ forms a precompact subset of $\mathbb{S}$. Therefore, we can conclude that $\mathcal{A}_2$ is compact.

Now, assume that $\{x_n\} \subset \aleph_\Phi$, with $x \in \mathbb{S}$ and $x_n \to x$ as $n \to \infty$, which implies that $|x_n(\varkappa) - x(\varkappa)| \to 0$ uniformly for $\varkappa \geq m(0)$ as $n \to \infty$. Because

$$\begin{aligned}
\left\|\mathcal{A}_2x_n-\mathcal{A}_2x\right\| &= \sup_{\varkappa\geq 0}\Big|\int_0^{\varkappa}\Psi(\varkappa,s)\Big\{\Big(2\mathcal{N}(s)r(s)-r'(s)\Big)\left[\pounds\left(x_n\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)\right.\\
&\quad-\pounds\left(x\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)\Big]-c(s)[\Gamma\left(x_n\left(s\right)+u^*\left(s\right)\right)\\
&\quad-\Gamma\left(x\left(s\right)+u^*\left(s\right)\right)]+c(s)r(s)[\Gamma\left(x_n\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)\\
&\quad-\Gamma\left(x\left(s-\wp(s)\right)+u^*\left(s-\wp(s)\right)\right)]\Big\}ds\Big|\\
&\leq \lambda\left[(2\left\|\mathcal{N}\right\|_*\left\|r\right\|+\delta_1)\,l+R\left\|c\right\|(1+\left\|r\right\|)\right]\left\|x_n-x\right\|,
\end{aligned}$$

coupled with the fact that Γ is continuous, it can be verified that $\|\mathcal{A}_2x_n - \mathcal{A}_2x\| \to 0$ as $n \to \infty$ and subsequently we can show that $\mathcal{A}_2$ exhibits continuous behavior.

(iii) For each $x, y \in \aleph_\Phi$, it follows that

$$\|\mathcal{B}_2 x - \mathcal{B}_2 y\| \leq l \|r\| \|x - y\|,$$

furthermore, recalling that $\|r\| \, l < 1$,it holds that $\mathcal{B}_2$ defines a contraction operator. Utilizing Krasnoselskii's fixed point theorem, there exists a function $\vartheta \in \aleph_\Phi$ satisfying $(\mathcal{A}_2 + \mathcal{B}_2)\,\vartheta = \vartheta$, making ϑ a valid solution of (3.28). Since the solution corresponding to Φ is unique, it follows that $v(\varkappa) \to 0$ as $\varkappa \to \infty$. □

Now we give an important result regarding the stability of the trivial solution, as expressed in the subsequent theorem.

Theorem 3.5. *If R and l satisfy*

$$1 - l \|r\| - \lambda \left(\left(2 \|\mathcal{N}\|_* \|r\| + \delta_1 \right) l + \|c\| \left(1 + \|r\| \right) R \right) > 0. \tag{3.40}$$

Thus, (3.1) has stable trivial solution.

Proof. Assuming that Γ and $\pounds$ meets the locally Lipschitz condition, D from Theorem 3.2 and $\gimel$ from Theorem 3.4 must exist, yielding constants $R, l > 0$ in a manner that

$$\left|\Gamma\left(v\left(\varkappa\right) + u^*\left(\varkappa\right)\right) - \Gamma(u^*(\varkappa))\right| < R \left|v\left(\varkappa\right)\right| \quad \text{and} \quad \left|\pounds\left(v\left(\varkappa\right) + u^*\left(\varkappa\right)\right) - \pounds(u^*(\varkappa))\right| < l \left|v\left(\varkappa\right)\right|.$$

Given that ϑ fulfills

$$\begin{aligned}
\vartheta\left(\varkappa\right) &= \Psi\left(\varkappa, 0\right)\left[\Phi(0) - r(0)\left(\pounds\left(\Phi\left(-\wp(0)\right) + u^*\left(-\wp(0)\right)\right) - H(u^*(-\wp(0)))\right)\right] \\
&\quad + r(\varkappa)\left(\pounds\left(\vartheta\left(\varkappa - \wp(\varkappa)\right) + u^*\left(\varkappa - \wp(\varkappa)\right)\right) - \pounds(u^*(\varkappa - \wp(\varkappa)))\right) + \int_0^t \Psi\left(\varkappa, s\right) \\
&\quad \times \Big\{ \Big(2\mathcal{N}(s) r(s) - r'(s)\Big)\left(H\left(\vartheta\left(s - \wp(s)\right) + u^*\left(s - \wp(s)\right)\right) - \pounds(u^*(s - \wp(s)))\right) \\
&\quad - c(s)\left(\Gamma\left(v\left(s\right) + u^*\left(s\right)\right) - \Gamma(u^*(s))\right) \\
&\quad + c(s) r(s)\left(\Gamma\left(\vartheta\left(s - \wp(s)\right) + u^*\left(s - \wp(s)\right)\right) - \Gamma(u^*(s - \wp(s)))\right)\Big\}\, ds,
\end{aligned}$$

then

$$\|\vartheta\| \leq \theta\left(1 + l\|r\|\right)\|\Phi\| + l\|r\|\|\vartheta\| + \lambda\left[\left(2\|\mathcal{N}\|_* \|r\| + \delta_1\right) l \|\vartheta\| + \|c\|\left(1 + \|r\|\right) R \|\vartheta\|\right],$$

that is

$$\left[1 - l\|r\| - \lambda\left(\left(2\|\mathcal{N}\|_* \|r\| + \delta_1\right) l + \|c\|\left(1 + \|r\|\right) R\right)\right]\|\vartheta\| \leq \theta\left(1 + l\|r\|\right)\|\Phi\|.$$

It is clear that there is a $\delta > 0$ for any $\xi > 0$ in a way that $\left|\vartheta\left(\varkappa\right)\right| < \xi$ for each $\varkappa \geq m\left(0\right)$ assuming $\|\Phi\| < \delta$, therefore the trivial solution of (3.28) exhibits stability. □

3.4 Application

In this section, we present a practical application that clearly and concretely illustrates the main results obtained in this study, particularly those concerning the existence and stability of periodic solutions for neutral differential systems with variable delays. This application aims to highlight the theoretical and practical significance of these results, especially in addressing problems related to electrical circuits and transmission lines, where such systems play a central role in modeling and analysis. Through this application, we demonstrate how the analytical methods employed in this work can be utilized to derive precise results that contribute to a deeper understanding of these complex systems and pave the way for their application in relevant practical contexts.

Example 3.3. *Consider the system of neutral differential equations*

$$\begin{aligned} &\frac{d}{d\varkappa}u(\varkappa) - r(\varkappa)\frac{d}{d\varkappa}\pounds(u(\varkappa - \varsigma(\varkappa))) = \mathcal{Q}(\varkappa) + \mathcal{N}(\varkappa)u(\varkappa) \\ &+ \mathcal{N}(\varkappa)r(\varkappa)\pounds(u(\varkappa - \varsigma(\varkappa))) - c(\varkappa)\Gamma(u(\varkappa)) + c(\varkappa)r(\varkappa)\Gamma(u(\varkappa - \varsigma(\varkappa))), \end{aligned} \tag{3.41}$$

in which $T = \frac{\pi}{2}$, $\varsigma(\varkappa) = 2\sin(4\varkappa)$, $r(\varkappa) = \frac{1}{140}\sin(4\varkappa)$, $c(\varkappa) = \sin(4\varkappa)$, $\mathcal{Q}(\varkappa) = \begin{pmatrix} 0 \\ 0.01\cos(4\varkappa) \end{pmatrix}$, $\mathcal{N}(\varkappa) = \begin{pmatrix} -\frac{1}{4}\cos 4\varkappa - \frac{3}{4} & -\frac{1}{4}\sin 4\varkappa - 1 \\ 1 - \frac{1}{4}\sin 4\varkappa & \frac{1}{4}\cos 4\varkappa - \frac{3}{4} \end{pmatrix}$,
$\Gamma(u(\varkappa)) = \begin{pmatrix} 0 \\ 0.1\cos(u(\varkappa)) \end{pmatrix}$, *and* $\pounds(u(\varkappa)) = \begin{pmatrix} \cos(u(\varkappa)) \\ 0 \end{pmatrix}$.

- ***Step1*** *In this section, we examine* ***the existence of a periodic solution****. For a sufficiently small positive value* $\mathbb{L} = 0.5$, *equation (3.41) can be rewritten as*

$$\begin{aligned} &\frac{d}{d\varkappa}v(\varkappa) - r_1(\varkappa)\frac{d}{d\varkappa}\pounds(v(\varkappa - \wp(\varkappa))) \\ &= \mathbb{L}\mathcal{Q}_1(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa) + \mathbb{L}\mathcal{N}_1(\varkappa)r_1(\varkappa)\pounds(v(\varkappa - \wp(\varkappa))) \\ &- \mathbb{L}c_1(\varkappa)\Gamma(v(\varkappa)) + \mathbb{L}c_1(\varkappa)r_1(\varkappa)\Gamma(v(\varkappa - \wp(\varkappa))), \end{aligned} \tag{3.42}$$

where $v(\varkappa) = u(0.5\varkappa)$, $\ell = \frac{T}{\mathbb{L}} = \pi$, $\wp(\varkappa) = \frac{\varsigma(0.5\varkappa)}{\mathbb{L}} = 4\sin(2\varkappa)$, $r_1(\varkappa) = \frac{1}{140}\sin(2\varkappa)$,
$c_1(\varkappa) = \sin(2\varkappa)$, $r_1(\varkappa) = \begin{pmatrix} 0 \\ 0.01\cos(2\varkappa) \end{pmatrix}$, $\mathcal{N}_1(\varkappa) = \begin{pmatrix} -\frac{3}{4} - \frac{1}{4}\cos 2\varkappa & -1 - \frac{1}{4}\sin 2\varkappa \\ 1 - \frac{1}{4}\sin 2\varkappa & -\frac{3}{4} + \frac{1}{4}\cos 2\varkappa \end{pmatrix}$,

We will calculate μ with

$$\begin{aligned}\mu &= \sup_{\varkappa\in[0,\ell]}\left(\sup_{\varkappa\leq s\leq\varkappa+\ell}\left|\left[\Upsilon(s)\left(\Upsilon^{-1}(\ell)-I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}\right|_*\right)\\ &= \sup_{\varkappa\in[0,\pi]}\left(\sup_{\varkappa\leq s\leq\varkappa+\pi}\left|\left[\Upsilon(s)\left(\Upsilon^{-1}(\ell)-I\right)\Upsilon^{-1}(\varkappa)\right]^{-1}\right|_*\right),\end{aligned}$$

We have

$$\Upsilon(\varkappa)=e^{\int_0^{\varkappa}\mathcal{N}_1(s)ds}=\begin{pmatrix}-e^{-\frac{\varkappa}{2}}\sin\varkappa & e^{-\varkappa}\cos\varkappa\\ e^{-\frac{\varkappa}{2}}\cos\varkappa & e^{-\varkappa}\sin\varkappa\end{pmatrix},$$

and

$$\Upsilon^{-1}(\varkappa)=\begin{pmatrix}-e^{\frac{1}{2}\varkappa}\sin\varkappa & e^{\frac{1}{2}\varkappa}\cos\varkappa\\ e^{\varkappa}\cos\varkappa & e^{\varkappa}\sin\varkappa\end{pmatrix}.$$

We find

$$\begin{aligned}&\left[\Upsilon(s)(\Upsilon^{-1}(\pi)-I)\Upsilon^{-1}(\varkappa)\right]^{-1}\\ &=\Upsilon(\varkappa)(\Upsilon^{-1}(\pi)-I)^{-1}\Upsilon^{-1}(s)\\ &=\begin{pmatrix}-e^{-\frac{\varkappa}{2}}\sin\varkappa & e^{-\varkappa}\cos\varkappa\\ e^{-\frac{\varkappa}{2}}\cos\varkappa & e^{-\varkappa}\sin\varkappa\end{pmatrix}\begin{pmatrix}0 & -0.04\\ -0.2 & 0\end{pmatrix}\begin{pmatrix}-e^{\frac{1}{2}s}\sin s & e^{\frac{1}{2}s}\cos s\\ e^{s}\cos s & e^{s}\sin s\end{pmatrix}\\ &=\begin{pmatrix}0.2e^{\frac{1}{2}s-\varkappa}\cos\varkappa\sin s+0.04e^{s-\frac{1}{2}\varkappa}\cos s\sin\varkappa & -0.2e^{\frac{1}{2}s-\varkappa}\cos s\cos\varkappa+0.04e^{s-\frac{1}{2}\varkappa}\sin s\sin\varkappa\\ 0.2e^{\frac{1}{2}s-\varkappa}\sin s\sin\varkappa-0.04e^{s-\frac{1}{2}\varkappa}\cos s\cos\varkappa & -0.2e^{\frac{1}{2}s-\varkappa}\cos s\sin\varkappa-0.04e^{s-\frac{1}{2}\varkappa}\cos\varkappa\sin s\end{pmatrix}\end{aligned}$$

Moreover, we have $\|\mathcal{N}_1\|_*=\sup_{\varkappa\in[0,\pi]}|\mathcal{N}|_*=\frac{9}{4}$, $\|r_1\|=\frac{1}{140}$, $\delta_1=\left\|r_1'\right\|=\frac{1}{70}$, $\gamma_2=\left\|1-\wp'(\varkappa)\right\|=5$, $|\pounds(0)|=1$, $l=1$ *and* $\eta_1=1+(1+\mathbb{L}\|\mathcal{N}_1\|_*)\mu\ell=64.62$. *The differential equation* $\frac{d}{d\varkappa}v(\varkappa)=\mathbb{L}\mathcal{N}_1(\varkappa)v(\varkappa)$ *is noncritical. Then setting* $D=15$ *and* $\rho=0.56$, *it follows that the condition of theorem 3.2 holds and thus (3.41) has a* $\frac{\pi}{2}$ *-periodic solution.*

• ***Step2*** ***[Asymptotic stability of periodic solution]****. Assigning* u^* *as a* $\frac{\pi}{2}$*-periodic solution (representing the equilibrium) for (3.41), and defining* $v(\varkappa)=u(\varkappa)-u^*(\varkappa)$*, system (3.41) becomes*

$$\begin{aligned}&\frac{d}{d\varkappa}v(\varkappa)-r(\varkappa)\frac{d}{d\varkappa}\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)\\ &=\mathcal{N}(\varkappa)v(\varkappa)+\mathcal{N}(\varkappa)r(\varkappa)\left(\pounds\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\pounds(u^*(\varkappa-\wp(\varkappa)))\right)\\ &-c(\varkappa)\left(\Gamma\left(v\left(\varkappa\right)+u^*\left(\varkappa\right)\right)-\Gamma(u^*(\varkappa))\right)\\ &+c(\varkappa)r(\varkappa)\left(\Gamma\left(v\left(\varkappa-\wp(\varkappa)\right)+u^*\left(\varkappa-\wp(\varkappa)\right)\right)-\Gamma(u^*(\varkappa-\wp(\varkappa)))\right).\end{aligned}\tag{3.43}$$

To determine the constants r and η, it can be shown that

$$\begin{aligned}
\lambda &= \sup_{\varkappa\geq 0}\left|\int_0^{\varkappa}\Psi(\varkappa,s)\,ds\right|_* = \sup_{\varkappa\geq 0}\left|\int_0^{\varkappa}\Theta(\varkappa)\,\Theta^{-1}(s)\,ds\right|_* \\
&= \sup_{\varkappa\geq 0}\left|\int_0^{\varkappa}\begin{pmatrix} -e^{-\frac{\varkappa}{2}}\sin\varkappa & e^{-\varkappa}\cos\varkappa \\ e^{-\frac{\varkappa}{2}}\cos\varkappa & e^{-\varkappa}\sin\varkappa \end{pmatrix}\begin{pmatrix} -e^{\frac{1}{2}s}\sin s & e^{\frac{1}{2}s}\cos s \\ e^{s}\cos s & e^{s}\sin s \end{pmatrix} ds\right|_* \\
&= \sup_{\varkappa\geq 0}\left|\int_0^{\varkappa}\begin{pmatrix} e^{s-\varkappa}\cos s\cos\varkappa + e^{\frac{1}{2}(s-\varkappa)}\sin s\sin\varkappa & e^{s-\varkappa}\cos\varkappa\sin s - e^{\frac{1}{2}(s-\varkappa)}\cos s\sin\varkappa \\ e^{s-\varkappa}\cos s\sin\varkappa - e^{\frac{1}{2}(s-\varkappa)}\cos\varkappa\sin s & e^{s-\varkappa}\sin s\sin\varkappa + e^{\frac{1}{2}(s-\varkappa)}\cos s\cos\varkappa \end{pmatrix} ds\right|_* \\
&= \frac{1}{20}\sup_{\varkappa\geq 0}\left|\begin{pmatrix} \left(9-10e^{-\varkappa}\cos\varkappa+\cos 2\varkappa-3\sin 2\varkappa+16(\sin\varkappa)\,e^{-\frac{1}{2}\varkappa}\right) \\ \left(13-10e^{-\varkappa}\sin\varkappa+3\cos 2\varkappa+\sin 2\varkappa-16(\cos\varkappa)\,e^{-\frac{1}{2}\varkappa}\right) \end{pmatrix}\right. \\
&\qquad \left.\begin{matrix} \left(10e^{-\varkappa}\cos\varkappa-13+3\cos 2\varkappa+\sin 2\varkappa+8(\sin\varkappa)\,e^{-\frac{1}{2}\varkappa}\right) \\ \left(9+10e^{-\varkappa}\sin\varkappa-\cos 2\varkappa+3\sin 2\varkappa-8(\cos\varkappa)\,e^{-\frac{1}{2}\varkappa}\right) \end{matrix}\right)\Bigg|_* \\
&= 3.7,
\end{aligned}$$

and

$$\begin{aligned}
\theta &= \sup_{\varkappa\geq 0}|\Psi(\varkappa,0)|_* = \sup_{\varkappa\geq 0}\left|\begin{pmatrix} -e^{-\frac{\varkappa}{2}}\sin\varkappa & e^{-\varkappa}\cos\varkappa \\ e^{-\frac{\varkappa}{2}}\cos\varkappa & e^{-\varkappa}\sin\varkappa \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right|_* \\
&= \sup_{\varkappa\geq 0}\left|\begin{pmatrix} (\cos\varkappa)\,e^{-\varkappa} & -(\sin\varkappa)\,e^{-\frac{1}{2}\varkappa} \\ (\sin\varkappa)\,e^{-\varkappa} & (\cos\varkappa)\,e^{-\frac{1}{2}\varkappa} \end{pmatrix}\right|_* \\
&= 2.
\end{aligned}$$

Thus, it implies the following

$$\Theta(\varkappa)\to 0 \ \text{as}\ \varkappa\to\infty,\ \varkappa-\varsigma(\varkappa)\to\infty,$$

In our case, we know that $D = 15$, $\rho = 0.56$, $l = 1$ and $R = 0.1$. Let $\daleth = 18$, it can be concluded that all conditions described by the above Theorems 3.4 and 3.5 have been shown to hold. Therefore the solution of (3.43) fulfills $v(\varkappa)\to 0$ as $\varkappa\to\infty$ and can be said to be stable, and so system (3.41) exhibits an asymptotically stable periodic solution

CONCLUSIONS AND PERSECTIVES

In this thesis, we have examined the asymptotic stability as well as existence of periodic solution for neutral differential system under specific sufficient conditions. Our method is underpinned by the fixed point method, along with the introduction of novel fixed mappings, leading us to define new conditions that guarantee such periodic solutions' existence and stability. The derived results build upon earlier studies by building on and generalizing prior work, particularly from neutral differential equations toward the wider context of neutral differential systems. This investigation highlights the effectiveness of Krasnoselskii's fixed point theorem for the analysis of periodicity and stability in systems that possess time delays. Through transforming the neutral differential system into a corresponding integral system while applying the fundamental matrix solution alongside Floquet theory, we have been able to define the conditions which ensure the existence of periodic solution. Finally, we have defined the conditions under which asymptotic stability can be guaranteed for these solutions, thereby ensuring their convergence toward zero over time. To demonstrate the applicability of these theoretical findings, practical examples are provided, validating the derived conditions and demonstrating their benefit in guaranteeing the stability and existence of periodic solution. In addition to improving the understanding of dynamical systems featuring time delays, this effort has the potential for wider application across areas like electrical circuit theory, control systems, and also biological modeling. In summary, this work provides a comprehensive framework to assist with the analysis of neutral differential systems, which offers new perspectives and helpful instruments to be used in investigating such systems as they appear across engineering and science.

In the future, the results of this thesis can be generalized to delayed neutral differential equations and systems in extended functional spaces such as L^p-spaces, Sobolav spaces, and in the time scale, as well as within fractional calculus. From the results discussed in this thesis, which primarily concern real-world models such as electric circuits, it is evident that the fixed-point method presented here holds promising prospects. On the other hand, there are still open research problems awaiting modification and development in fixed-point theory, such as relaxing the Lipschitz condition and reformulating the convexity condition of the set in a suitable way that allows solving such problems. Our hope is very high in achieving these promising prospects.

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