وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

BADJI MOKHTAR-ANNABA UNIVERSITY

UNIVERSITE BADJI MOKHTAR ANNABA

جامعة باجي مختار
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THESIS

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On a class of sequential fractional differential equations with delay

Stream
Applied Mathematics

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By
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THÈSE

En vue de l'obtention du diplôme de doctorat

Sur une classe d'équations différentielles fractionnaires séquentielles à retard

Filière
Mathématiques Appliquées

Spécialité
Mathématiques Appliquées
Par

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حول فئة من المعادلات التفاضلية الكسرية المتسلسلة ذات التأخير

ملخّص

ندرس في هذه الرسالة المعادلات التفاضلية الكسرية التسلسلية مع التأخيرات غير المنتهية، بعضها مرتبط بشروط ابتدائية، والبعض الآخر مرتبط بشروط حدية. نقوم بتحويل كل مشكلة معينة إلى مشكلة نقطة ثابتة ثم نستخدم نظريات النقطة الثابتة للحصول على وجود وتفرد الحل. ندرس عدة أنواع مختلفة من الشروط الحدية، مثل الشروط الحدية التكاملية الثلاثية والشروط الحدية الطبيعية. بالنسبة لمسائل الشروط الحدية غير المحلية، نقدم أيضا تحليلًا للمشكلة بدون تأخير عند الرنين. علاوة على ذلك، نقدم نتائج جديدة لتحليل مسائل القيم الحدية الكسرية متعددة الحدود مع شروط حدية كسرية.

كلمات مفتاحية: المسائل الحدية، المشتقات الكسرية ،وجود الحلول، نظريات النقطة الثابتة.

On a class of sequential fractional differential equations with delay

Abstract

We study in this thesis sequential fractional differential equations with delay, some of which are combined with initial conditions, and others are associated with boundary conditions. We transform every given problem into a fixed point problem and then use fixed point theorems to obtain the existence and the uniqueness of the solution. In the boundary value problems that we examine, different types of boundary conditions are inspected, such as three-point integral boundary conditions and natural boundary conditions. In addition to our investigations of non local boundary value problems, we present an analysis for the problem without delay at resonance. Moreover, we furnish new results for analysing multi-term fractional boundary value problems with fractional boundary conditions.

Keywords: Boundary value problems ; Fractional derivatives ; Existence of solutions ; Fixed point theorems.

Sur une classe d'équations différentielles fractionnaires séquentielles à retard

Résumé

Nous étudions dans cette thèse des équations différentielles fractionnaires séquentielles à retard. Certaines sont combinées avec des conditions initiales, et d'autres sont associées à des conditions aux limites. Nous transformons chaque problème donné en un problème de point fixe, puis nous utilisons des théorèmes de point fixe pour obtenir l'existence et l'unicité de la solution. Dans les problèmes aux limites que nous examinons, différents types de conditions aux limites sont étudiés, telles que les conditions intégrales à trois points et les conditions naturelles. Pour les problèmes aux limites non locaux, nous présentons aussi une analyse pour le problème sans retard à résonance. De plus, nous fournissons de nouveaux résultats pour l'analyse des problèmes aux limites fractionnaires multitermes avec des conditions aux bords fractionnaires.

Mots-clés : Problèmes aux limites ; Dérivées fractionnaires ; Existence de la solution ; Théorèmes du point fixe.

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Contents

Introduction

The analysis of differential equations of the fractional type is prevalent in the current mathematical research. Effectively, it has been proved in [1] and numerous other papers that many real-world problems are rather of the fractional type. Hence, fractional derivatives extend the integer ones, providing better accuracy when studying these problems. Fractional derivatives differ greatly from their classical opponents. Namely, in the fractional setting there exist numerous definitions of fractional derivatives, which represents an ongoing challenge but also provides a certain level of freedom when choosing the adequate definition regarding the associated initial or boundary conditions. Another difference resides in the appearance of a singular kernel, which is advantageous as it emphasises the nonlocal feature of fractional derivatives. For this reason, fractional differential equations enhance the capturing, understanding, and interpretation of physical behaviours, particularly in processes characterised by memory and delay.

Nonlocal conditions are better suited to model certain physical problems that depend on internal points of the domain, which makes them more appropriate than the standard boundary conditions. Furthermore, integral boundary conditions play a crucial role in describing many processes, such as cellular systems, blood fluid, and population dynamics. However, the presence of these explicit non localities hinders the study of sequential differential equations.

Insofar as their nonlocal property is concerned, when modelling fractional differential equations mathematically, the delay effect must be considered. While the literature on delay fractional differential equations is plethoric, results concerning the fractional boundary value problems with delay are not covered fully.

Furthermore, higher order differential equations result from the repeated composition of first order differential operators. They represent a direct consequence of the semigroup property of the integer order derivative, i.e., $\underbrace{D \circ \cdots \circ D}_{\text{n times}} u = D^n u$. However, fractional derivatives are not commutative; therefore, they do not enjoy the semigroup property. Hence, the composition of two or more fractional differential operators of lower orders gives rise to the notion of sequential derivatives. Accordingly, sequential fractional derivatives emerge naturally from physical situations in which derivatives are taken consecutively. Therefore, generalisations of certain applied higher order differential problems to the fractional case must be replaced in a natural manner by sequential fractional derivatives, as these differential problems are necessarily of the sequential type.

The impact of the inclusion of sequential derivatives in the differential problems covered in this thesis resides mainly in the technical aspects of our proofs. Whereas, when pursuing the existence of positive solutions, one can see that the influence of sequential fractional derivatives is more pervasive on the problem analysis. Precisely, one is faced with the challenge of changing conventional methods to overcome the complications arising from the appearance of at least two parameters when contemplating to establish a Harnack inequality for the associated Green's function. This investigation elevates the mathematical insights of the considered problem. This promising direction is still in its early stages, but we do not pursue it here.

Our goal is to establish sufficient conditions so that the above mentioned class of equations will possess solutions. Nonetheless, we cannot tackle all problems involving fractional derivatives with the same mathematical techniques. In most cases, we employ the standard process of converting the differential equation into an integral equation. Then we apply fixed point theory to the corresponding operator. However, in some other cases, it is challenging to furnish an analytical form of the solution. In the latter case, topological methods are rather effective.

The roadmap for this thesis is provided below.

In Chapter 1, we recall the main tools relevant to our investigations. Our novel results in this area are then covered in the remaining four subsequent chapters.

In Chapter 2, we begin by highlighting the motivation for our study; specifically, we describe sequential and delay differential equations. Then we formulate our research question as an initial value problem. We establish the theoretical foundations, then introduce appropriate function spaces. We inspect some fundamental existence and uniqueness results. Once existence has been established, we seek the stability analysis via the Hyers-Ulam stability theorem. Progressively, throughout this chapter, we impose weaker conditions on the nonlinearity in order to obtain the intended results. These results are the focus of the joint paper [2] with Guezane-Lakoud Assia and Khaldi Rabah. This paper was published in the journal Nonlinear Dynamics and Systems Theory in February 2024.

In Chapter 3, we examine a more complicated fractional problem consisting of a sequential delay differential equation and three-point integral boundary conditions. Specifically, we investigate the solvability of the given problem under various restrictions on the nonlinearity. This chapter is derived from the publication titled "Existence Results for Delay Three-point Fractional Boundary Value Problems," which was submitted in January 2025.

In Chapter 4, we study the resonance case of the preceding problem without delay. Precisely, we inspect an existence result for the given differential equation. Due to the complexity of the problem, it is not possible to construct an analytical form of the solution as before. We overcome this hurdle by employing Mawhin's degree theory. Thus, we begin by providing the main ingredients that constitute the basis for the proof of our main result. The final section illustrates the applicability of the results obtained via an example. This chapter is the result of the joint work with Guezane-Lakoud Assia and Khaldi Rabah, titled "Existence of Solutions for a Three-point Sequential Caputo Boundary Value Problem at Resonance." This paper was submitted in April 2025.

In Chapter 5, we investigate the existence of solutions for a novel multi-term fractional differential problem. In this final task, we initiate the study of a Riemann-Liouville-Caputo fractional boundary value problem entailing natural fractional boundary conditions. These results are taken from the paper titled "Analysis of RLC Multi-term Fractional Boundary Value Problems," which was submitted jointly with Guezane-Lakoud Assia and Khaldi Rabah in September 2024.

Finally, we conclude this manuscript by outlining some future work.

Chapter 1

Preliminaries

In this chapter, fundamental tools and theoretical requisites are presented briefly; namely, some elements of fractional calculus and some rudimentary definitions and results from the theory of functional analysis are provided.

1.1 Some space functions and special functions

1.1.1 Absolutely continuous functions

Definition 1.1. *[3] We say that a function $u:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, if for every $\varepsilon>0$ there exists $\delta>0$ such that for every set of pairwise disjoint subintervals $[a_i,b_i]$ of $[a,b]$, $i=1,2,\cdots,n$,*

$$\sum_{i=1}^{n}(b_i-a_i)<\delta \implies \sum_{i=1}^{n}|u(b_i)-u(a_i)|<\varepsilon.$$

The notation $AC([a,b])$ stands for the following space:

$$\{u:[a,b]\to\mathbb{R};\, u \text{ is absolutely continuous}\}.$$

This space is characterized by the following assertion.
$u\in AC([a,b])$ if and only if $u(t)=c+\int_a^t v(s)ds$, where $\int_a^b|v(s)|ds<\infty$.

Definition 1.2. *Let $n\in\mathbb{N}$. The space of absolutely continuous functions of order n is defined by*

$$AC^n([a,b])=\{u\in C^{n-1}([a,b]),\, u^{(n-1)}\in AC([a,b])\}.$$

Lemma 1.1. *The space* $AC^n([a,b])$ *comprises only the following functions.*

$$u(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}\phi(s)ds + \sum_{k=0}^{n-1} c_k(t-a)^k$$

with $\phi \in L^1([a,b])$, *given by* $\phi(t) = u^{(n)}(t)$, *and* $c_k = \frac{u^{(k)}(a)}{k!}$.

1.1.2 Some special functions

The gamma function It is a well-known special function that extends the factorial to positive numbers.

Definition 1.3. *Define the gamma function as*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \;\; \alpha > 0.$$

It satisfies the following recurrence relationship.

Lemma 1.2.

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \text{ for every } \alpha > 0.$$

Proof. Integrating by parts, we find

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}dx = [-x^\alpha e^{-x}]_0^\infty + \alpha\int_0^\infty x^{\alpha-1}e^{-x}dx = \alpha\Gamma(\alpha).$$

□

Employing the previous lemma, we infer that $\Gamma(n+1) = n!$, $n \in \mathbb{N}$.

Let $\alpha > 0$ be a positive number. Let n be a given integer. We have the following property:

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1). \tag{1.1}$$

We present another special function that is connected to the gamma function.

The beta function

Definition 1.4. *Define the beta function as*

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\ p>0,\ q>0.$$

A relation between the beta and the gamma functions We begin by writing the beta function under its trigonometric form.
Setting $x=sin^2\theta$ gives $dx=2sin\theta\ cos\theta\ d\theta$, so that

$$\begin{aligned}B(p,q)&=\int_0^{\frac{\pi}{2}}(sin^2\theta)^{p-1}(cos^2\theta)^{q-1}(2cos\theta\ sin\theta)d\theta\\&=2\int_0^{\frac{\pi}{2}}(sin\theta)^{2p-1}(cos\theta)^{2q-1}d\theta.\end{aligned}$$

Proposition 1.1. *For every $p>0$ and $q>0$,*

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Proof. Set $x=y^2$, we get $dx=2y\,dy$, so that

$$\Gamma(p)=2\int_0^\infty y^{2p-1}e^{-y^2}dy,$$

$$\Gamma(q)=2\int_0^\infty x^{2p-1}e^{-x^2}dx.$$

Then using the polar coordinates, we evaluate the double integral:

$$\begin{aligned}\Gamma(p)\Gamma(q)&=4\int_0^\infty\int_0^\infty x^{2q-1}y^{2p-1}e^{-(x^2+y^2)}dxdy\\&=4\int_0^\infty\int_0^{\frac{\pi}{2}}(rcos\theta)^{2q-1}(rsin\theta)^{2p-1}e^{-r^2}rd\theta dr\\&=4\int_0^\infty r^{2(p+q)-1}e^{-r^2}dr\int_0^{\frac{\pi}{2}}(cos\theta)^{2q-1}(sin\theta)^{2p-1}d\theta\\&=\Gamma(p+q)B(p,q).\end{aligned}$$

□

1.2 Theory of fractional calculus

This section concerns fractional integration and differentiation. We introduce the fractional integral of Riemann-Liouville. Moreover, we give three different definitions of fractional derivatives. Namely, the Riemann-Liouville, the Caputo, and the Riemann-Liouville-Caputo fractional derivatives are discussed. Several results serving as a stepping stone for the subsequent chapters are furnished below; see [3–6].

1.3 The Cauchy formula for integration

Let $f \in L^1([a,b])$. The first primitive of f is defined as

$$I^1 u(t) = \int_a^t u(s)ds.$$

The second primitive of f is obtained as follows:

$$I^2 u(t) = \int_a^t \int_a^\tau u(s)dsd\tau.$$

Using Fubini's theorem, we get

$$I^2 u(t) = \int_a^t \int_s^t u(s)d\tau ds = \int_a^t (t-s)u(s)ds.$$

Iterating the above process n times, we find

$$I^n u(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u(s)ds.$$

This iterated integration is called the Cauchy formula. By employing $(n-1)! = \Gamma(n)$, we extend the preceding formula to define the fractional Riemann-Liouville integral.

1.3.1 The Riemann-Liouville fractional integral

Definition 1.5. *Let* $u \in L^1([a,b])$, $\alpha > 0$. *The left Riemann-Liouville (RL) fractional integral is defined by*

$$I_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t u(s)(t-s)^{\alpha-1} ds, \; t > a,$$

and the right one is defined as

$$I_{b-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b u(s)(t-s)^{\alpha-1} ds, \; t < b.$$

Now, we give the fractional analogue of the integration by parts identity.

Proposition 1.2. *Let* $u, v \in L^1([a,b])$, *therefore;*

$$\int_a^b u(t)(I_{a+}^{\alpha} v(t))dt = \int_a^b v(t)(I_{b-}^{\alpha} u(t))dt.$$

Proof. By virtue of the Fubini theorem,

$$\begin{aligned}
\int_a^b u(t)(I_{a+}^{\alpha} v(t))dt &= \int_a^b u(t) \frac{1}{\Gamma(\alpha)} \int_a^t v(s)(t-s)^{\alpha-1} ds dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t (t-s)^{\alpha-1} u(t) v(s) ds dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b v(s) \int_s^b u(t)(t-s)^{\alpha-1} dt ds \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b v(s)(I_{b-}^{\alpha} u(s)) ds.
\end{aligned}$$

□

Proposition 1.3. *Let* $\alpha > 0$, $\beta > -1$; *hence,*

$$I_{a+}^{\alpha}\left((s-a)^{\beta}\right)(t) = \Gamma(\beta+1) \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}. \tag{1.2}$$

Proof. We have

$$I^{\alpha}((s-a)^{\beta})(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^{\beta}ds.$$

Set $\tau=\frac{s-a}{t-a}$, then $ds=(t-a)d\tau,\ \ s-a=\tau(t-a),\ \ t-s=(t-a)(1-\tau)$. Moreover, when $s=a,\quad \tau=0$, and when $s=t,\ \ \tau=1$.

$$\begin{aligned}
I_{a+}^{\alpha}((s-a)^{\beta})(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (t-a)^{\alpha-1}(1-\tau)^{\alpha-1}\tau^{\beta}(t-a)^{\beta}(t-a)d\tau \\
&= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta}d\tau \\
&= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}B(\alpha,\beta+1) \\
&= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Gamma(\beta+1).
\end{aligned}$$

□

We have the following property, called the semigroup property.

Proposition 1.4. *Let* $\phi\in L^1([a,b]),\ \alpha,\beta>0$. *Therefore, for almost every* $t\in[a,b]$,

$$I_{a+}^{\alpha}I_{a+}^{\beta}u(t)=I_{a+}^{\alpha+\beta}u(t),\quad I_{b-}^{\alpha}I_{b-}^{\beta}u(t)=I_{b-}^{\alpha+\beta}u(t). \tag{1.3}$$

Proof.

$$\begin{aligned}
I_{a+}^{\alpha}I_{a+}^{\beta}u(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}I_{a+}^{\beta}u(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\frac{1}{\Gamma(\beta)}\int_a^s u(\tau)(s-\tau)^{\beta-1}d\tau ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^s (t-s)^{\alpha-1}u(\tau)(s-\tau)^{\beta-1}d\tau ds.
\end{aligned}$$

Reversing the order of integration, owing to Fubini's theorem,

$$\begin{aligned} I^{\alpha}_{a+}I^{\beta}_{a+}u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_\tau^t u(\tau)(t-s)^{\alpha-1}(s-\tau)^{\beta-1}dsd\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t u(\tau)\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1}dsd\tau. \end{aligned}$$

Set $s=\tau+x(t-\tau)$ so that $ds=(t-\tau)dx$ and $t-s=(t-\tau)(1-x)$. When $s=\tau$, $x=0$, and when $s=t$, we get $x=1$. Hence,

$$\begin{aligned} \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1}ds &= \int_0^1 (t-\tau)^{\alpha-1}(1-x)^{\alpha-1}x^{\beta-1}(t-\tau)^{\beta-1}(t-\tau)dx \\ &= (t-\tau)^{\alpha+\beta-1}\int_0^1 x^{\beta-1}(1-x)^{\alpha-1}dx. \end{aligned}$$

Using

$$\int_0^1 x^{\beta-1}(1-x)^{\alpha-1}dx = B(\alpha,\beta) = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)},$$

yields

$$\begin{aligned} I^{\alpha}_{a+}I^{\beta}_{a+}u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t u(\tau)(t-\tau)^{\alpha+\beta-1}B(\alpha,\beta)d\tau. \\ &= \frac{1}{\Gamma(\beta+\alpha)}\int_a^t u(\tau)(t-\tau)^{\alpha+\beta-1}d\tau \\ &= I^{\alpha+\beta}_{a+}u(t). \end{aligned}$$

For the right-sided integral, we proceed likewise. □

Next we give another property of the RL fractional integral, whose proof relies on classical theorems on integration.

Proposition 1.5. *Let $\alpha>0$. The operators $I^{\alpha}_{a+}, I^{\alpha}_{b-}$ map continuous functions into continuous functions.*

1.3.2 The Riemann-Liouville fractional derivative

Definition 1.6. *Let $0 < \alpha < 1$. The left Riemann-Liouville (RL) fractional derivative of order α is given by*

$$(D^{\alpha}_{a+}u)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}u(s)ds,$$

and the right one is

$$(D^{\alpha}_{b-}u)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (t-s)^{-\alpha}u(s)ds.$$

We give, in the subsequent result, a sufficient condition for the RL fractional derivative to exist.

Lemma 1.3. *Let $0 < \alpha < 1$. Then $D^{\alpha}_{a+}u$ and $D^{\alpha}_{b-}u$ exist almost everywhere on $[a,b]$ provided that $u \in AC([a,b])$. Furthermore, $D^{\alpha}_{a+}u$, $D^{\alpha}_{b-}u \in L^r([a,b])$, $1 \leq r < \frac{1}{\alpha}$ and*

$$(D^{\alpha}_{a+}u)(t) = \frac{1}{\Gamma(1-\alpha)}(\frac{u(a)}{(t-a)^{\alpha}} + \int_a^t (t-s)^{-\alpha}u'(s)ds),$$

$$(D^{\alpha}_{b-}u)(t) = \frac{1}{\Gamma(1-\alpha)}(\frac{u(b)}{(b-t)^{\alpha}} - \int_t^b (t-s)^{-\alpha}u'(s)ds).$$

Next, we define the Riemann-Liouville fractional derivative for $\alpha > 0$.

Definition 1.7. *If $\alpha \in \mathbb{N}$, then the fractional RL derivatives and the classical derivatives match as follows:*

$$D^{\alpha}_{a+} = (\frac{d}{dt})^{\alpha},\ D^{\alpha}_{b-} = (-\frac{d}{dt})^{\alpha}.$$

Definition 1.8. *The left and right RL fractional derivatives of order α are given by the following expressions:*

$$(D^{\alpha}_{a+}u)(t) = (\frac{d}{dt})^{[\alpha]}(D^{\{\alpha\}}_{a+}u)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_a^t (t-s)^{n-\alpha-1}u(s)ds,\ t > a,$$

$$(D^{\alpha}_{b-}u)(t) = (\frac{d}{dt})^{[\alpha]}(D^{\{\alpha\}}_{b-}u)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_t^b (s-t)^{n-\alpha-1}u(s)ds,\ t < b,$$

where $n=[\alpha]+1$, *and* $\{\alpha\}=\alpha-[\alpha]$ *is the fractional part of* α.

Example 1.1. *Let* $\alpha>0$, $\beta>-1$, *then* $(D^{\alpha}_{a+}(t-a)^{\beta})(x)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}$.

Proof. Indeed, let $n=[\alpha]+1$, in view of (1.2),

$$\begin{aligned}(D^{\alpha}_{a+}(s-a)^{\beta})(t)&=(\frac{d}{dt})^n[I^{n-\alpha}_{a+}(t-a)^{\beta}]\\&=(\frac{d}{dt})^n[\frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}(t-a)^{n-\alpha+\beta}]\\&=\frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}(\frac{d}{dt})^n(t-a)^{n-\alpha+\beta}.\end{aligned}$$

Given $(\frac{d}{dt})^n(t-a)^{n-\alpha+\beta}=(n-\alpha+\beta)(n-\alpha+\beta-1)(n-\alpha+\beta-2)\cdots(1-\alpha+\beta)t^{\beta-\alpha}$, and employing (1.1), we obtain

$$\begin{aligned}(D^{\alpha}_{a+}(s-a)^{\beta})(t)&=\frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}\frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}\\&=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.\end{aligned}$$

□

Theorem 1.1. *Let* $\alpha>0$, $u\in AC^n([a,b])$, $n=[\alpha]+1$. *Thus,* $D^{\alpha}_{a+}u$ *is defined almost everywhere and*

$$D^{\alpha}_{a+}u(t)=\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}+\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{\alpha-n+1}u^{(n)}(s)ds.$$

Proof. Since $u\in AC^n([a,b])$, by virtue of Lemma 1.1, we have

$$\begin{aligned}D^{\alpha}_{a+}u(t)&=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_a^t(t-s)^{n-\alpha-1}\Big(\frac{1}{(n-1)!}\int_a^s(s-\tau)^{n-1}u^{(n)}(s)d\tau\\&+\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\Big)ds\\&=\frac{1}{\Gamma(n-\alpha)(n-1)!}(\frac{d}{dt})^n\int_a^t\int_a^s(t-s)^{n-\alpha-1}(s-\tau)^{n-1}u^{(n)}(\tau)d\tau ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}\int_a^t (t-s)^{n-\alpha-1}(s-a)^k ds\\
&=I_1+I_2.
\end{aligned}
$$

Inverting the order of integration in I_1 and then applying the same change of variable from the proof of formula (1.3), we find

$$
\begin{aligned}
I_1 &= \frac{1}{\Gamma(n-\alpha)(n-1)!}(\frac{d}{dt})^n\int_a^t u^{(n)}(\tau)(t-\tau)^{2n-\alpha-1}B(n,n-\alpha)ds\\
&=\frac{1}{\Gamma(2n-\alpha)}(\frac{d}{dt})^n\int_a^t u^{(n)}(s)(t-\tau)^{2n-\alpha-1}d\tau.
\end{aligned}
$$

By differentiating n times,

$$
\begin{aligned}
I_1 &= \frac{\Gamma(2n-\alpha)}{\Gamma(2n-\alpha)\Gamma(n-\alpha)}\int_a^t u^{(n)}(\tau)(t-\tau)^{n-\alpha-1}d\tau\\
&=\frac{1}{\Gamma(n-\alpha)}\int_a^t u^{(n)}(\tau)(t-\tau)^{n-\alpha-1}d\tau.
\end{aligned}
$$

For the second term I_2, from the proof of (1.2) we have

$$
\int_a^t (t-s)^{n-\alpha-1}(s-a)^k ds=(t-a)^{n-\alpha+k}B(k+1,n-\alpha)=(t-a)^{n-\alpha+k}\frac{k!\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)}.
$$

Thus, employing $(\frac{d}{dt})^n t^\delta=\frac{\Gamma(\delta+1)}{\Gamma(\delta-n+1)}t^{\delta-n}$, we get

$$
\begin{aligned}
I_2 &= \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{\Gamma(k+1+n-\alpha)}(\frac{d}{dt})^n(t-a)^{n-\alpha+k}\\
&=\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{\Gamma(n-\alpha+k+1)}\frac{\Gamma(n+k-\alpha+1)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}\\
&=\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}.
\end{aligned}
$$

□

Lemma 1.4. *Let* $\phi\in L^1([a,b])$, $\alpha>0$. *The Abel's integral equation* $I^\alpha_{a+}\phi=0$ *admits the* *unique trivial solution* $\phi \equiv 0$, *almost everywhere on* $[a,b]$.

Theorem 1.2. *Let* $\alpha > 0$, $u \in L^1([a,b])$. *Therefore, for almost every* $t \in [a,b]:$

1. $D^{\alpha}_{a+} I^{\alpha}_{a+} u(t) = u(t)$.
2. *We have*
$$I^{\alpha}_{a+} D^{\alpha}_{a+} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} u_{n-\alpha}^{(n-k-1)}(a), \tag{1.4}$$
where $n = [\alpha]+1$ *and* $u_{n-\alpha} = I^{n-\alpha}_{a+} u$.

Proof. 1. We have
$$(D^{\alpha}_{a+} I^{\alpha}_{a+}) u(t) = (\frac{d}{dt})^n I^{n-\alpha} I^{\alpha} u(t).$$
By the semigroup property (1.3), we get
$$(D^{\alpha}_{a+} I^{\alpha}_{a+}) u(t) = (\frac{d}{dt})^n I^{n} u(t).$$
By the theory of classical differentiation we obtain
$$(D^{\alpha}_{a+} I^{\alpha}_{a+}) u(t) = u(t).$$

2. Since $u_{n-\alpha} = I^{n-\alpha}_{a+} u \in AC^n([a,b])$, then through Lemma 1.1, it can be represented as
$$I^{n-\alpha}_{a+} u(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u^{(n)}_{n-\alpha}(s) ds + \sum_{k=0}^{n-1} \frac{u^{(k)}_{n-\alpha}(a)}{k!} (t-a)^k.$$
Set $j = n-k-1$. Hence,
$$\begin{aligned} I^{n-\alpha}_{a+} u(t) &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u^{(n)}_{n-\alpha}(s) ds + \sum_{j=0}^{n-1} \frac{u^{(n-j-1)}_{n-\alpha}(a)}{(n-j-1)!} (t-a)^{n-j-1} \\ &= I^{n}_{a+} D^{\alpha}_{a+} u + \sum_{k=0}^{n-1} \frac{u^{(n-k-1)}_{n-\alpha}(a)}{\Gamma(n-k)} (t-a)^{n-k-1} \\ &= I^{n}_{a+} D^{\alpha}_{a+} u + \sum_{k=0}^{n-1} \frac{I^{n-\alpha}_{a+} (t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} u^{(n-k-1)}_{n-\alpha}(a), \end{aligned}$$

where we used

$$I_{a+}^{n-\alpha}(t-a)^{\alpha-k-1} = \frac{(t-a)^{n-k-1}\Gamma(\alpha-k)}{\Gamma(n-k)}.$$

Hence,

$$I_{a+}^{n-\alpha}[u - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} u_{n-\alpha}^{(n-k-1)}(a)] = I_{a+}^{n-\alpha}(I_{a+}^{\alpha} D_{a+}^{\alpha} u).$$

By using Lemma 1.4, we deduce that

$$u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} u_{n-\alpha}^{(n-k-1)}(a) = I_{a+}^{\alpha} D_{a+}^{\alpha} u.$$

□

1.3.3 The Caputo fractional derivative

Definition 1.9. *Assume that the left and right RL fractional derivatives of order α exist. Then we define the left Caputo fractional derivative of order α by*

$$({}^{c}D_{a+}^{\alpha} u)(t) = (D_{a+}^{\alpha}[u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(s-a)^{k}])(t), \quad n = [\alpha]+1.$$

The right Caputo fractional derivative of the same order is defined as

$$({}^{c}D_{b-}^{\alpha} u)(t) = (D_{b-}^{\alpha}[u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(b-s)^{k}])(t), \quad n = [\alpha]+1.$$

Remark 1.1. *Zero is the fractional Caputo derivative of any constant.*

Example 1.2. *Let $\alpha > 0$, $\beta \geq 0$.*

$$({}^{c}D_{a+}^{\alpha}(s-a)^{\beta})(t) = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, & \beta \in \mathbb{R},\ \beta > [\alpha],\ or\ \beta \in \mathbb{N},\ \beta \geq [\alpha]+1, \\ 0, & \beta \in \mathbb{N},\ \beta \leq [\alpha]. \end{cases} \tag{1.5}$$

Remark 1.2. *If* $0 < \alpha < 1$, *then*

$$({}^cD^{\alpha}_{a+}u)(t) = (D^{\alpha}_{a+}[u(s) - u(a)])(t),$$

$$({}^cD^{\alpha}_{b-}u)(t) = (D^{\alpha}_{b-}[u(s) - u(b)])(t).$$

Proposition 1.6. *If* $D^{\alpha}_{a+}u$ *and* ${}^cD^{\alpha}_{a+}u$ *exist. Then*

$$({}^cD^{\alpha}_{a+}u)(t) = (D^{\alpha}_{a+}u)(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}.$$

We have the same result for the right fractional derivative.

Proof. Owing to the definition and

$$D^{\alpha}_{a+}(\sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(s-a)^k)(t) = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha},$$

we retrieve the result based on the linearity of the RL fractional derivative.

□

Remark 1.3. *1. For* $0 < \alpha < 1$, *we have*

$${}^cD^{\alpha}_{a+}u(t) = D^{\alpha}_{a+}u(t) - \frac{u(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}.$$

2. The Caputo fractional derivative exists once the RL derivative exists, in particular, whenever $u \in AC^n([a,b])$, *as shown in the preceding section.*

Theorem 1.3. *Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $n = [\alpha] + 1$, $u \in AC^n([a,b])$. *The left and right Caputo derivatives of order* α *exist for almost every* $t \in [a,b]$. *Moreover,*

$$({}^cD^{\alpha}_{a+}u)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds,$$

$$({}^cD^{\alpha}_{b-}u)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1}u^{(n)}(s)ds.$$

Proof. By definition,

$$
\begin{aligned}
({}^cD^{\alpha}_{a+}u)(t) &= (D^{\alpha}_{a+}[u(s)-\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k])(t)\\
&= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_a^t (t-s)^{n-\alpha-1}[u(s)-\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k]ds.
\end{aligned}
$$

Set $\frac{dw}{ds}=(t-s)^{n-\alpha-1}$, so that $w(s)=-\frac{(t-s)^{n-\alpha}}{(n-\alpha)}$.
Set $v(s)=u(s)-\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k$, integrating by parts yields

$$
\begin{aligned}
({}^cD^{\alpha}_{a+}u)(t) &= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n[[-\frac{(t-s)^{n-\alpha}}{(n-\alpha)}v(s)]_a^t+\int_a^t\frac{(t-s)^{n-\alpha}}{(n-\alpha)}v'(s)ds]\\
&= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_a^t\frac{(t-s)^{n-\alpha}}{n-\alpha}v'(s)ds\\
&= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^{n-1}\int_a^t(t-s)^{n-\alpha-1}v'(s)ds.
\end{aligned}
$$

Repeating the same process $n-1$ times then using $v^{(n)}(s)=u^{(n)}(s)$ gives

$$
({}^cD^{\alpha}_{a+}u)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}u^{(n)}(s)ds.
$$

□

Lemma 1.5. *Let* $u\in AC^n([a,b])$; *therefore,*

$$
{}^cD^{\alpha}_{a+}I^{\alpha}_{a+}u(t)=u(t),\ a.e.\ \ on\,[a,b]. \tag{1.6}
$$

Corollary 1.1. *Let* $u\in AC^n([a,b])$, *thus*

$$
{}^cD^{\alpha}_{a+}I^{\beta}_{a+}u(t)=I^{\beta-\alpha}_{a+}u(t),\ \alpha<\beta,\ a.e.\ \ on\,[a,b]. \tag{1.7}
$$

Theorem 1.4. *For every* $u\in AC^n([a,b])$, $\alpha>0$,

$$
I^{\alpha}_{a+}\,{}^cD^{\alpha}_{a+}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k.
$$

Proof. Owing to the semigroup property of the RL integral and in view of (1.2), we have

$$\begin{aligned} I^{\alpha}_{a+}\, {}^{c}D^{\alpha}_{a+}u(t) &= I^{\alpha}_{a+} I^{n-\alpha}_{a+} D^{n}_{a+}u(t) \\ &= I^{n}_{a+} D^{n}_{a+}u(t) \\ &= u(t) - \sum_{k=0}^{n-1} \frac{u^{(n-k-1)}(a)}{\Gamma(n-k)}(t-a)^{n-k-1}. \\ &= u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t-a)^{k}. \end{aligned}$$

□

Corollary 1.2. *Let $\alpha > 0$. The equation ${}^{c}D^{\alpha}_{a+}u = 0$ possesses the unique solution*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},\ c_i \in \mathbb{R},\ i = 0,1,2,\cdots,n-1.$$

1.3.4 The Riemann-Liouville-Caputo fractional derivative

Definition 1.10. *Let $1 < \alpha < 2$, the Riemann-Liouville-Caputo fractional derivative of order α is given as ${}^{RLC}D^{\alpha}_{a+}u(t) = \frac{d}{dt}\,{}^{c}D^{\alpha-1}_{a+}u(t)$, whenever it exists.*

A relationship between this fractional derivative and the previous ones is given below.

Proposition 1.7. *Let $1 < \alpha < 2$. If $u'(t)$ is absolutely continuous on $[a,b]$, ${}^{c}D^{\alpha}_{a+}u(t)$ and $D^{\alpha}_{a+}u(t)$ exist. Moreover,*

$$D^{\alpha}_{a+}u(t) = {}^{RLC}D^{\alpha}_{a+}u(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}u(a),$$

and

$${}^{c}D^{\alpha}_{a+}u(t) = {}^{RLC}D^{\alpha}_{a+}u(t) - \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}u'(a).$$

The proof relies on integration by parts, and it is straightforward; therefore, it is omitted.

Remark 1.4. *Note that the above differential and integral operators are linear. Moreover, their definitions extend to the complex case, which is not covered in this work. In what follows, we denote* I^{α}_{0+} *by* I^{α}, D^{α}_{0+} *by* D^{α}, ${}^{c}D^{\alpha}_{0+}$ *by* ${}^{c}D^{\alpha}$ *and* ${}^{RLC}D^{\alpha}_{0+}$ *by* ${}^{RLC}D^{\alpha}$.

1.3.5 The Sequential fractional derivative

The sequential fractional differential operator, introduced for the first time by Miller and Ross in their book [7], is denoted $\mathscr{D}^{\sigma_k}$ and defined by $\mathscr{D}^{\sigma_k} = D^{\alpha_k}D^{\alpha_{k-1}}\cdots D^{\alpha_1}$, where $\sigma_0 = 0$, $\sigma_k = \sum_{j=1}^{k}\alpha_j, (k = 1,2,\cdots,n), \quad 0 < \alpha_j < 1$, $(j = 1,2,\cdots,n)$, and D^{α} is a specified fractional derivative.

1.4 Delayed fractional differential equations

1.4.1 Nonlinear delayed fractional differential equations

We call a nonlinear fractional delay differential equation every equation referred to as such:

$$D^{\alpha}u(t) = f(t,u(t-\theta)),\ t \in [0,b],$$

where D^{α} is a specified fractional derivative of order α and $f : \mathbb{R}\times C([-r,0],\mathbb{R})$ is the nonlinearity. When $\theta = 0$, the equation is called an ordinary fractional differential equation with no delay; when $\theta < 0$, the equation is called an advanced one; and when $\theta > 0$, the equation is called a delayed or retarded one. We are interested in the latter. Let $\theta < 0$ be a constant. We denote by $u_t : [-r,0] \to \mathbb{R}$ the delayed unknown function given by $u_t(\theta) = u(t-\theta)$, $\theta \in [-r,0]$ with $r \geq 0$. Further, we say that the delay is finite if the constant r is finite. We call the above equation a fractional differential equation with infinite delay when r is infinite. Hereafter, we consider r to be infinite.

1.4.2 Phase spaces

Infinite delay differential equations are associated with history functions as follows:

$$\begin{cases} D^{\alpha}u(t) = f(t,u_t), & t \in (0,b], \\ u(t) = \phi(t), & t \in (-\infty,0] \end{cases} \tag{1.8}$$

with $\phi \in \mathbb{B}$ being the history function and the phase space $\mathbb{B}$ is defined below.
We introduce and use the following space when the fractional differential equation contains an infinite delay. In order to examine delay differential equations, an axiomatic construction of convenient spaces, called phase spaces, can be found in several books [8–10]. A phase space $\mathbb{B}$ is a semi-normed linear space of functions $\phi : \mathbb{R}_- \to \mathbb{R}$ which is characterised by three fundamental axioms listed below.
For $u : (-\infty,b] \to \mathbb{R}$ such that $u_0 \in \mathbb{B}$, we have for all $t \in [0,b]$

1. $u_t \in \mathbb{B}$,
$$\|u_t\|_{\mathbb{B}} \leq K(t)\sup\{|u(s)| : 0 \leq s \leq t\} + M(t)\|u_0\|_{\mathbb{B}}, \tag{1.9}$$
$$|u(t)| \leq H\|u_t\|_{\mathbb{B}}$$
with H being a positive constant; $K \in C([0,b],\mathbb{R}_+)$; $M : \mathbb{R}_+ \to \mathbb{R}_+$ is locally bounded; H,K and M do not depend on $u(.)$,

2. the function $u_t : [0,b] \to \mathbb{B}$ is continuous,

3. $\mathbb{B}$ is a complete space.

1.5 Some Results From Operator Theory

This section encompasses the fundamental tools and results that constitute the basis for the proofs of the forthcoming chapters. We define the notions of contraction, compactness, and relative compactness, and then we state the Arzelà-Ascoli theorem. Additionally, we provide some fixed point theorems, Mawhin's degree theory, and a modified Grönwall lemma.

Definition 1.11 (Contraction mapping). *Let $(X, \|.\|)$ be a normed space. $K \subset X$ a closed subset of X. We say that a mapping $\mathscr{T} : K \to K$ is a contraction if there is $k \in (0,1)$ verifying*

$$\|\mathscr{T}x - \mathscr{T}y\| \leq k\|x-y\|, \quad \text{for every}\, x,y \in K. \tag{1.10}$$

Definition 1.12. *(Compactness) Let $\mathscr{M}$ be a nonempty subset of X. We say that $\mathscr{M}$ is compact if for every open covering of $\mathscr{M}$, $\mathscr{M} \subset \bigcup_{i\in I} O_i$, there exist finitely many indices $\{i_1, i_2, \dots, i_k\}$ such that $\mathscr{M} \subset \bigcup_{i=1}^{i=k} O_{i_k}$.*

Definition 1.13. *(Relative compactness) We say that a nonempty subset $\mathscr{M}$ of X is relatively compact if its closure $\overline{\mathscr{M}}$ is compact.*

Definition 1.14 (A compact operator). *We say that $\mathscr{T} : X \to Y$ is a compact operator if $\mathscr{T}(B_X)$ is relatively compact in Y.*

Theorem 1.5. ***(Arzelà-Ascoli)*** *[11] Let $(X, \|.\|)$ be a normed space and $\mathscr{M} \subset X$ a compact subset of X. Then a subset $\mathscr{F}$ of $C(\mathscr{M}, X)$ is relatively compact in X if and only if*

1. *for all $x \in \mathscr{M}$, $\{f(x), f \in \mathscr{F}\}$ is a relatively compact set in X,*

2. *$\mathscr{F}$ is equicontinuous, that is, for any $\varepsilon > 0$, there exists a $\delta > 0$ so that for all $x, y \in \mathscr{M}$, we have $\|x-y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon$, for every $f \in \mathscr{M}$.*

1.5.1 Some standard fixed point theorems

Theorem 1.6. ***(Banach contraction principle)*** *[11] Let X be a Banach space. Let $\mathscr{A} : X \to X$ be a contraction. Hence, $\mathscr{A}$ admits a unique fixed point.*

Theorem 1.7. ***(Leray-Schauder nonlinear alternative)*** *[12] Let X be a Banach space, $\mathscr{U}$ an open subset of X, $0_E \in \mathscr{U}$, and $\mathscr{A} : \overline{\mathscr{U}} \to X$ a continuous and compact operator. Consequently, one of the following assertions holds:*

1. *$\mathscr{A}$ admits a fixed point $x^* \in \overline{\mathscr{U}}$, or*

2. *there exists $x \in \partial\mathscr{U}$, $0 < \lambda < 1$ satisfying $x = \lambda\mathscr{A}x$.*

Theorem 1.8. ***(Krasnoselskii's fixed point theorem)*** *[11] Let $\mathscr{M}$ be a nonempty, bounded, closed, and convex subset of a Banach space X. Let $\mathscr{A}, \mathscr{B} : \mathscr{M} \to X$ be two operators such that*

1. *$\mathscr{A}(z_1) + \mathscr{B}(z_2) \in \mathscr{M}$, for all $z_1, z_2 \in \mathscr{M}$,*

2. *$\mathscr{A}$ is continuous and compact,*

3. *$\mathscr{B}$ is a contraction,*

therefore; $\mathscr{A} + \mathscr{B}$ admits a fixed point $z \in \mathscr{M}$.

1.5.2 Mawhin's coincidence degree theory

In this subsection, we introduce a topological method that does not depend on studying the form of the explicit solution, called the Mawhin's coincidence degree theory, see [13]. Let $(X, \|.\|_X), (Y, \|.\|_Y)$ be real Banach spaces. Let $\mathscr{U}$ be an open bounded subset of X, $L\colon domL \subset X \to Y$ a linear operator, and $N\colon \overline{\mathscr{U}} \to Y$ a nonlinear operator. We are interested in the equation $Lx = Nx$.

Definition 1.15. *[13] A linear mapping $L\colon domL \subset X \to Y$ is said to be a Fredholm operator of index zero if*

1. *ImL is a closed subset of Y,*

2. *$dimKerL = codimImL < +\infty$.*

We infer from definition 1.15 the existence of continuous projections $P\colon X \to X$, $Q\colon Y \to Y$ satisfying $ImP = KerL$, $KerQ = ImL$, $X = KerL \oplus KerP$, and $Y = ImL \oplus ImQ$. It follows that the restriction of L to $domL \cap KerP$ is an isomorphism onto ImL and its inverse is denoted by $K_P\colon ImL \to domL \cap KerP$.

Definition 1.16. *[13] Let L be a Fredholm operator of index zero. Let $\mathscr{U}$ be an open bounded subset of X such that $domL \cap KerP \neq \emptyset$. We say that operator $N\colon \overline{\mathscr{U}} \to Y$ is L-compact if*

1. *operator $QN\colon \overline{\mathscr{U}} \to Y$ is continuous and $QN(\overline{\mathscr{U}}) \subseteq Y$ is bounded,*
2. *the operator $K_P(I-Q)N\colon \overline{\mathscr{U}} \to X$ is completely continuous.*

Theorem 1.9. *[13] Let $L\colon domL \subset X \to Y$ be a Fredholm operator of index zero. Let $N\colon X \to Y$ be L-compact on $\overline{\mathscr{U}}$. Assume the following assumptions hold:*

1. *$Lx \neq \mu Nx, (x,\mu) \in (domL \setminus KerL) \cap \partial\mathscr{U} \times (0,1)$,*
2. *$Nx \notin ImL, x \in KerL \cap \partial\mathscr{U}$,*
3. *$deg(QN\big|_{KerL}, \mathscr{U} \cap KerL, 0) \neq 0$.*

Hence, the equation $Lx = Nx$ admits a solution in $domL \cap \overline{\mathscr{U}}$.

1.5.3 Hyers-Ulam stability

[14] We shall now introduce and define the Hyers-Ulam stability and the generalised Hyers-Ulam stability. Let $(X, \|.\|)$ be a Banach space. Let $\mathscr{A}\colon X \to X$ be an operator. Consider the equation

$$x = \mathscr{A}x, \;\; x \in X \tag{1.11}$$

and the inequality

$$\|y - \mathscr{A}y\| \leq \varepsilon, \;\; y \in X, \tag{1.12}$$

where $\varepsilon > 0$ is a constant.

Definition 1.17. *We say that problem (1.11) is Hyers-Ulam stable if there exists* $\lambda > 0$, *for any* $\varepsilon > 0$, *and for every* y *solution of (1.12), there exists* $x \in X$ *solution of (1.11) such that* $\|x - y\| \leq \lambda \varepsilon$.

Definition 1.18. *We say that Problem(1.11) is generalised Hyers-Ulam stable if there is a function* $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $\varphi(0) = 0$ *so that for every* $y \in X$ *solution for (1.12), there exists* $x \in X$ *solution for (1.11) where* $\|x - y\| \leq \varphi(\varepsilon)$.

Remark 1.5. *The Hyers-Ulam stability implies the generalized Hyers-Ulam stability.*

1.5.4 Some inequalities

A modified Grönwall Lemma

Lemma 1.6. *[15] Let* $v : [0,b] \to \mathbb{R}_+$ *be a real function, let* $w(.)$ *be a nonnegative, locally integrable function on* $[0,b]$, *and there exist constants* $R > 0$ *and* $0 < \alpha < 1$ *with*

$$v(t) \leq w(t) + R \int_0^t \frac{v(s)}{(t-s)^{\alpha}} ds, \; t \in [0,b].$$

Hence, a constant $K = K(\alpha)$ *exists, for which*

$$v(t) \leq w(t) + KR \int_0^t \frac{w(s)}{(t-s)^{\alpha}} ds, \; t \in [0,b].$$

Chapter 2

On Sequential Riemann-Liouville Initial Value Problems with Infinite Delay

2.1 Introduction

Fractional differential equations are spreading widely to affect various fields of science; see [16, 17]. Effectively, they adequately model numerous physical processes, particularly those involving memory and delay. The latter is justified by the nonlocal characteristic of fractional derivatives, which allows the consideration of the past states of the function, resulting in their efficiency when studying delayed differential models compared to the classical ones.

Delay differential equations in the classical sense of derivation have been examined extensively; their theoretical aspects have been largely investigated; see [18–22]. Moreover, there exist many papers corresponding to real world models that are marked with delay, namely neural networks, population dynamics, ecological models, and disease models; see [23]. Nevertheless, fractional delay differential equations are still not yet fully addressed.

Unlike the integer order case, fractional derivatives are non commutative. As a consequence, in the fractional setting, one cannot convert multi-term fractional differential equations into systems of one-term equations. Nonetheless, this lack of commutativity gives rise to sequential fractional derivatives. While being introduced for the first time by Dzherbashian and Neresian in [24], a Russian paper, translated into English only in

2020, they are currently known as the Miller Ross sequential derivatives for being studied in their monograph [7]. Recently, sequential fractional derivatives have received more attention in literature, given their significance; see [25–27]. Effectively, they are easily spotted in physics, where it is frequent to substitute formulas containing derivatives for one another. Since sequential fractional derivatives appear naturally, it is necessary to explore further their different features.

Below, we briefly outline some substantial works that are concerned with sequential fractional derivatives and infinite delay.

In 2012, Furati [28] studied the following Riemann-Liouville sequential fractional differential equation.

$$D^{\alpha}[(t-a)^{r}D^{\beta}u(t)] = f(t,u),\ t \in (0,b].$$

In 2020, Fazli et al. [29] studied a general Basset-Boussinesq-Oseen fractional equation and proved the global existence-uniqueness and regularity of solutions in a partially ordered Banach space.

$$\begin{aligned}
&D^{\alpha}(D^{\beta}+A)u(t)+Bu(t) = f(t),\ \ 0<t\leq 1,\ 0<\alpha,\beta\leq 1,\\
&u(0)=a,\\
&D^{\beta}u(0)=b.
\end{aligned}$$

In 2020, Fazli et al. [30] studied the Langevin fractional sequential differential equation and established the existence and the uniqueness results, employing Banach and Weissinger fixed point theorems:

$$\begin{aligned}
&{}^{c}D^{\beta}({}^{c}D^{\alpha}+\lambda)u(t) = f(t,u(t)),\ t\in(0,1],\\
&{}^{c}D^{i}u(0)=\mu_i,\ 0\leq i<l,\\
&{}^{c}D^{i}({}^{c}D^{\alpha}u)(0)=\nu_i,\ 0\leq i<n,
\end{aligned}$$

$m-1<\alpha\leq m,\ n-1<\beta\leq n,\ l=\max(n,m).$

While the literature on both infinite delay fractional initial value problems and sequential fractional derivatives is rich, their simultaneous presence is largely overlooked. Based on our best knowledge, fractional differential problems of sequential Riemann-Liouville type that are associated with infinite delay have not yet been examined. The primary focus of this chapter is to initiate the former study. Specifically, we pursue sufficient conditions to ensure the existence, the uniqueness, and the stability of solutions for the infinite delay IVP:

$$\begin{aligned} &D^{\beta}D^{\alpha}u(t) = f(t,u_t),\ t\in(0,b],\\ &D^{\alpha}u(0)=0,\\ &u(t)=\phi(t), t\in(-\infty,0], \end{aligned} \tag{2.1}$$

where $0<\alpha,\beta<1$, $f:[0,b]\times\mathbb{B}\to\mathbb{R}$, $\phi\in\mathbb{B}$, $\phi(0)=0$, $u_t(\theta)=u(t+\theta)$, $\theta\leq 0$. We recall that the phase space $\boldsymbol{B}$, which is already introduced in Chapter 1, is a semi-normed linear space of functions mapping $(-\infty,0]$ into $\mathbb{R}$. It is constructed axiomatically and used to treat differential equations with infinite delay. Further detailed inspections of unbounded delay differential equations can be found in [8–10]. Our results further enhance the literature concerning this specific class of fractional differential equations. By considering a novel situation consisting of an initial value problem (IVP) having a sequential Riemann-Liouville fractional derivative and delay, we emphasise that our findings are novel in this particular context.

This chapter's remaining sections are organised in the following manner: The second section is concerned with the preliminary result that is needed in our work. In the third section, we prove the existence of solutions. In Section 4, we address the stability analysis of the given problem in order to emphasise the physical meaning of our findings. Finally, with the purpose of highlighting the viability of our outcomes, a concrete example is provided in Section 5.

2.2 Existence Results

Proposition 2.1. *Let* $0 < \alpha < 1$. *Let* $F : [0,b] \to \mathbb{R}$ *be continuous. Therefore,* u *is a solution of the IVP*

$$\begin{aligned} &D^{\alpha}u(t) = F(t),\ t \in (0,b], \\ &u(0) = 0. \end{aligned} \tag{2.2}$$

if and only if $u \in C[0,b]$ *is a solution for the equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s)ds. \tag{2.3}$$

Proof. Let $u \in C[0,b]$ such that $D^{\alpha}u = F$, i.e., $DI^{1-\alpha}u = F$. Integrating, we get $I^{1-\alpha}u(t) = I^{1-\alpha}u(0) + I^1F$, so that $I^{1-\alpha}u$ is absolutely continuous. Apply operator I^{α} to the differential equation in (2.2), then by virtue of (1.4), we get $u(t) = \frac{c}{\Gamma(\alpha)}t^{\alpha-1} + I^{\alpha}F(t)$. Employing the initial condition, we find $c = 0$. Hence, (2.3) holds.
Conversely, if $u = I^{\alpha}F$, then, taking into account proposition 1.5, we see that u is continuous. Moreover, $u(0) = I^{\alpha}F(0) = 0$ because F is continuous. Then applying operator $I^{1-\alpha}$ and (1.3), we obtain $I^{1-\alpha}u = I^1F$. Differentiating, we get $D^{\alpha}u = F$. □

We shall provide under sufficient conditions on the nonlinearity an existence-uniqueness result through the Banach contraction theorem. Furthermore, by replacing the Lipschitz condition by a growth condition, we prove an existence result through the Leray-Schauder nonlinear alternative. Consider the following space: $\Omega = \{u : (-\infty, b] \to \mathbb{R},\ u_0 \in \mathbb{B}$, and $u_{\big|[0,b]}$is continuous$\}$.

Definition 2.1. *We call a solution for* (2.1) *every function* $u \in \Omega$, *which verifies the fractional differential equation* $D^{\beta}D^{\alpha}u(t) = f(t,u_t)$ *on* $(0,b]$ *and the initial conditions* $D^{\alpha}u(0) = 0$, $u(t) = \phi(t)$ *on* $(-\infty, 0]$.

Lemma 2.1. *Let $F(t) = f(t,u_t) \in C[0,b]$. Then u is a solution for the IVP:*

$$\begin{aligned} &D^{\beta} D^{\alpha} u(t) = f(t,u_t),\ t \in (0,b],\\ &D^{\alpha} u(0) = 0, \\ &u(0) = 0 \end{aligned} \tag{2.4}$$

if and only if u is a solution for the equation

$$u(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s) ds.$$

Proof. Let u be a solution for the IVP (2.4). Since F is continuous, then using proposition 2.1 yields

$$\begin{aligned} &D^{\alpha} u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,u_s) ds,\\ &u(0) = 0. \end{aligned}$$

Since $I^{\beta} F$ is also continuous, applying proposition 2.1 once again, we find

$$\begin{aligned} &u(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s) ds,\\ &u(0) = 0. \end{aligned}$$

We show the converse in a similar way. □

2.2.1 Uniqueness result based upon the Banach contraction principle

Theorem 2.1. *Assume $f : [0,b] \times \mathbb{B} \to \mathbb{R}$ is continuous Lipschitzian, that is, there exists an $L > 0$ so that*

$$|f(t,u) - f(t,v)| \leq L \|u - v\|_{\mathbb{B}}, \quad \textit{for every } t \in [0,b],\ u,v \in \mathbb{B},$$

where

$$\frac{b^{\alpha+\beta}K_bL}{\Gamma(\alpha+\beta+1)} < 1, \tag{2.5}$$

and $K_b = \sup_{t\in[0,b]} |k(t)|$. *Then the IVP (2.1) admits a unique solution on* $[0,b]$.

Proof. By means of the previous lemma, we prove that solving the IVP is the same as showing that the operator $S:\Omega\to\Omega$ admits a unique fixed point, where

$$(Su)(t) = \begin{cases} \phi(t), & t\in(-\infty,0], \\ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u_s)ds, & t\in[0,b]. \end{cases} \tag{2.6}$$

Consider the following decomposition.
Define $x(.) : (-\infty,b]\to\mathbb{R}$ as the function given by

$$x(t) = \begin{cases} 0, & t\in[0,b], \\ \phi(t), & t\in(-\infty,0]. \end{cases} \tag{2.7}$$

Let $v(.) : (0,b]\to\mathbb{R}$ and $\overline{v}$ be the function defined as

$$\overline{v}(t) = \begin{cases} v(t), & t\in[0,b], \\ 0, & t\in(-\infty,0]. \end{cases} \tag{2.8}$$

Take u to be a solution for the fixed point problem (2.6). We partition u into $u=\overline{v}+x$, $t\in[0,b]$, then $u_t = \overline{v}_t + x_t$, $t\in[0,b]$.
In addition, set $C_0 = \{v\in C([0,b]) : v_0 = 0\}$ equipped with the semi-norm in C_0 defined by $\|v\|_b = \sup\{|v(t)|,\ 0\le t\le b\}$. Consider $T : C_0\to C_0$, the operator defined as

$$(Tv)(t) = \begin{cases} 0, & t\in(-\infty,0], \\ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(\tau,\overline{v}_s + x_s)ds, & t\in[0,b]. \end{cases} \tag{2.9}$$

Clearly, stating that S possesses a fixed point is the same as stating that T admits a fixed point.

T is a contraction. Consider $v, v^* \in C_0$, then for any $t \in [0,b]$,

$$\begin{aligned}
|(Tv)(t)-(Tv^*)(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,\overline{v}_s+x_s)-f(s,\overline{v_s^*}+x_s)|ds \\
&\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}L\|\overline{v}_s-\overline{v_s^*}\|_{\mathbb{B}}ds \\
&\leq \frac{LK_b}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\sup_{\tau\in[0,s]}|v(\tau)-v^*(\tau)|ds \\
&\leq \frac{LK_b}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\|v-v^*\|_b ds \\
&\leq \frac{LK_b b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|v-v^*\|_b.
\end{aligned}$$

By means of (2.5), T is indeed a contraction mapping. Consequently, employing the Banach contraction theorem, we see that T admits a unique fixed point. □

2.2.2 Existence result based upon Leray-Schauder nonlinear alternative

Here, we present an existence result that relies on the nonlinear alternative of Leray-Schauder.

Theorem 2.2. *Let f be continuous and verify the following assumption:*
There exist $p, q \in C([0,b], \mathbb{R}_+)$ with

$$|f(t,u)| \leq p(t)+q(t)\|u\|_{\mathbb{B}}, \quad t\in[0,b],\ u\in\mathbb{B}.$$

Then the IVP (2.1) *admits a solution on* $[0,b]$.

Proof. Let $T : C_0 \to C_0$ be defined as in (2.9). We shall prove that T is a continuous and a completely continuous operator.
We claim that T is continuous. Indeed, let (v_n) be a sequence in C_0 with $v_n \to v$ in C_0.

Thus,

$$\begin{aligned}|(Tv_n)(t)-(Tv)(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^b (t-s)^{\alpha+\beta-1}|f(s,\overline{v}_{n_s}+x_s)-f(s,\overline{v}_s+x_s)|ds\\ &\leq \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|f(.,\overline{v}_{n_{(.)}}+x_{(.)})-f(.,\overline{v}_{(.)}+x_{(.)})\|_\infty,\end{aligned}$$

which tends to zero when n tends to $+\infty$.

Next, T is uniformly bounded in C_0. Let $v\in B_\eta := \{v\in C_0 : \|v\|_b\leq\eta\}$. Then for any $t\in[0,b]$,

$$\begin{aligned}|(Tv)(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^b (t-s)^{\alpha+\beta-1}|f(s,\overline{v}_s+x_s)|ds\\ &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^b (t-s)^{\alpha+\beta-1}(p(s)+q(s)\|\overline{v}_s+x_s\|_{\mathbb{B}})ds\\ &\leq \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty+\frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|q\|_\infty\eta_* =: l,\end{aligned}$$

where $\|\overline{v}_s+x_s\|_{\mathbb{B}}\leq\|\overline{v}_s\|_{\mathbb{B}}+\|x_s\|_{\mathbb{B}}\leq K_b\eta+M_b\|\phi\|_{\mathbb{B}} := \eta_*$. Hence, $\|Tv\|_\infty\leq l$.

Now, we show that T is uniformly equicontinuous on C_0. Let $0\leq t_1<t_2\leq b$. Take B_η to be defined as above and $v\in B_\eta$, then

$$\begin{aligned}|(Tv)(t_2)-(Tv)(t_1)| &\leq \frac{1}{\Gamma(\alpha+\beta)}|\int_0^{t_1}[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}]f(s,\overline{v}_s+x_s)ds\\ &+\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}f(s,\overline{v}_s+x_s)ds|\\ &\leq \frac{1}{\Gamma(\alpha+\beta)}(\int_0^{t_1}|(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}||f(s,\overline{v}_s+x_s)|ds\\ &+\int_{t_1}^{t_2}|(t_2-s)^{\alpha+\beta-1}f(s,\overline{v}_s+x_s)|ds)\\ &\leq \frac{\|p\|_\infty+\|q\|_\infty\eta_*}{\Gamma(\alpha+\beta)}\int_0^{t_1}|(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}|ds\\ &+\frac{\|p\|_\infty+\|q\|_\infty\eta_*}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}|(t_2-s)^{\alpha+\beta-1}|ds.\end{aligned}$$

which approaches 0, as t_1 tends to t_2. Owing to the previous steps and the Arzelá-Ascoli theorem, we infer that $T: C_0\to C_0$ is continuous and completely continuous.

Now, it is sufficient to prove the existence of an open set $U \subseteq C_0$ with $v \neq \lambda T v$, $\lambda \in (0,1)$, $v \in \partial U$. By contradiction, take $v \in C_0$ with $v = \lambda T(v)$, $0 < \lambda < 1$. Then for every $t \in [0,b]$,

$$\begin{aligned}|v(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1}(p(\tau)+q(\tau)\|\bar{v}_\tau + x_\tau\|_{\mathbb{B}})d\tau \\ &\leq \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} q(\tau)\|\bar{v}_\tau + x_\tau\|_{\mathbb{B}} d\tau.\end{aligned}$$

Then $\|\bar{v}_\tau + x_\tau\|_{\mathbb{B}} \leq K_b \sup\{|v(s)|,\ 0 \leq s \leq \tau\} + M_b\|\phi\|_{\mathbb{B}} := w(\tau)$, which implies that

$$|v(t)| \leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} q(\tau) w(\tau) d\tau + \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty,\ t \in [0,b].$$

By inserting the above in w, we find for every $t \in [0,b]$,

$$w(t) \leq M_b\|\phi\|_{\mathbb{B}} + \frac{K_b b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty + \frac{K_b\|q\|_\infty}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha-1}(t-\tau)^\beta w(\tau) d\tau.$$

Subsequently, applying Lemma 1.6 yields the existence a constant $K = K(\alpha)$ so that for each $t \in [0,b]$,

$$|w(t)| \leq M_b\|\phi\|_{\mathbb{B}} + \frac{K_b b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty + K(\alpha)\frac{K_b\|q\|_\infty b^\beta}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha-1} R\, d\tau,$$

where

$$R = M_b\|\phi\|_{\mathbb{B}} + \frac{K_b b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty.$$

Hence,

$$\|w\|_\infty \leq R + R\frac{K(\alpha)K_b b^{\alpha+\beta}}{\alpha\Gamma(\alpha+\beta)}\|q\|_\infty := r.$$

Then $\|v\|_\infty \leq r\|I^{\alpha+\beta}q\|_\infty + \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|p\|_\infty := r^*$.
Set $\mathscr{U} = \{v \in C_0 : \|v\|_b < r^* + 1\}$. $T : \overline{\mathscr{U}} \to C_0$ is continuous and completely continuous. Based on the choice of $\mathscr{U}$, there is no $v \in \partial\mathscr{U}$ satisfying $v = \lambda T(v)$, for $\lambda \in (0,1)$.
To this end, the nonlinear alternative of Leray-Schauder is applicable. Consequently, T admits a fixed point v in $\mathscr{U}$. □

2.3 Stability Analysis

In this section, we pursue a stability result by inspecting sufficient conditions under which the initial value problem (2.1) is Hyers-Ulam stable.

Definition 2.2. *We say that problem (2.1) is Hyers-Ulam stable if there exists a constant* $\lambda > 0$ *so that for every* $\varepsilon > 0$, *for every function* $u \in \Omega$ *solution for the problem*

$$\begin{aligned} &|D^{\beta} D^{\alpha} u - f(t, u_t)| \leq \varepsilon,\ t \in [0, b], \\ &u(t) = \phi(t),\ t \in (-\infty, 0], \end{aligned} \tag{2.10}$$

the IVP (2.1) possesses a solution $v \in \Omega$ *with* $|u(t) - v(t)| \leq \lambda \varepsilon,\ t \in [0, b]$.

Theorem 2.3. *Suppose that* $f : [0, b] \times \to \mathbb{R}$ *is continuous and Lipschitzian with the Lipschitz constant* L *satisfying (2.5). Then the IVP (2.1) is Hyers-Ulam stable.*

Proof. Since the conditions of Therem 2.1 are satisfied the unique solution $v \in \Omega$ to (2.1) takes the form

$$v(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, v_s) ds.$$

Now, let u be a solution to (2.10), then there exists a function w with $|w(t)| \leq \varepsilon$ and

$$D^{\beta} D^{\alpha} u = f(t, u_t) + w(t).$$

Proceeding as in Section 2, we find

$$u(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, u_s) ds + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} w(s) ds,$$

which gives

$$\begin{aligned} |u(t) - \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, u_s) ds| &\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |w(s)| ds \\ &\leq \frac{\varepsilon}{\Gamma(\alpha+\beta)} b^{\alpha+\beta} := \Lambda \varepsilon. \end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned}|u(t)-v(t)| &\le |u(t)-\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,v_s)ds| \\ &\le |u(t)-\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,u_s)ds| \\ &+\frac{1}{\Gamma(\alpha+\beta)}|\int_0^t (t-s)^{\alpha+\beta-1}f(s,u_s)-\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,v_s)ds| \\ &\le \Lambda\varepsilon+\frac{L}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha-1}(t-s)^{\beta}\|u_s-v_s\|_B ds \\ &\le \Lambda\varepsilon+\frac{Lb^{\beta}K_b}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha-1}\sup_{0\le\tau\le s}|u(\tau)-v(\tau)|ds.\end{aligned} \tag{2.12}$$

Set $g(t)=\sup_{0\le\tau\le t}|u(\tau)-v(\tau)|$, so

$$g(t)\le \Lambda\varepsilon+\frac{Lb^{\beta}K_b}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha-1}g(s)ds.$$

Applying the Grönwall lemma 1.6, we find

$$\begin{aligned}g(t) &\le \Lambda\varepsilon+\frac{Lb^{\beta}K_b}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha-1}\Lambda\varepsilon ds \\ &\le \Lambda\varepsilon+K\frac{LK_b}{\alpha}\Lambda^2\varepsilon=(\Lambda+K\frac{LK_b}{\alpha}\Lambda^2)\varepsilon.\end{aligned} \tag{2.13}$$

2.4 Example

Here, we show the viability of our outcomes through a numerical example. Take $\alpha=\beta=\frac{1}{2}$, $b=1$, $\gamma>0$. The nonlinearity $f:[0,1]\times\mathbb{B}_{\gamma}\to\mathbb{R}$, defined as

$$f(t,x)=e^{-\gamma t}(\frac{1}{\sqrt{t+4}}\frac{x^2+2|x|}{1+|x|}+\sin(t))$$

and $\chi \in \mathbb{B}_\gamma$, which is defined by

$$\mathbb{B}_\gamma = \{\chi : C((-\infty,0],\mathbb{R}) : \lim_{s\to-\infty} e^{\gamma s}|\chi(s)| \text{exists in } \mathbb{R}\},$$

equipped with the norm $\|\chi\|_{\mathbb{B}_\gamma} = \sup_{s\in(-\infty,0]} e^{\gamma s}|\chi(s)|$. It is easily verified that $\mathbb{B}_\gamma$ is an admissible phase space, i.e., it is a Banach space, and it fulfills the phase space axioms with $K(t) = 1$, $M(t) = e^{-\gamma t}$ and $H = 1$.

Moreover, for any $t \in [0,1]$, $x,y \in \mathbb{B}_\gamma$, we have

$$\begin{aligned}
|f(t,x) - f(t,y)| &= e^{-\gamma t}\frac{1}{\sqrt{t+4}}\left|\frac{x^2+2|x|}{1+|x|} - \frac{y^2+2|y|}{1+|y|}\right| \\
&\leq e^{-\gamma t}\frac{1}{\sqrt{t+4}}\left|\frac{x^2-y^2}{(1+|x|)(1+|y|)}\right| \\
&\leq e^{-\gamma t}\frac{1}{\sqrt{t+4}}|x-y| \leq \frac{1}{2}\|x-y\|_{\mathbb{B}_\gamma},
\end{aligned}$$

so that f satisfies the Lipschitz condition with $L = \frac{1}{2}$ and $\frac{b^{\alpha+\beta}K_b L}{\Gamma(\alpha+\beta+1)} = \frac{1}{4} < 1$, so the IVP has exactly one solution by virtue of Theorem 2.1. Also, the hypothesis from Theorem 2.3 is verified; hence, the IVP is shown to be Hyers-Ulam stable.

Furthermore, for any $t \in [0,1]$ and $x \in \mathbb{B}_\gamma$,

$$|f(t,x)| \leq \frac{e^{-\gamma t}}{\sqrt{t+4}}\frac{2+|x|}{1+|x|}|x| + e^{-\gamma t}\sin(t) \leq p(t)\|x\|_{\mathbb{B}_\gamma} + q(t),$$

where $p(t) = \frac{2}{\sqrt{t+4}}$ and $q(t) = e^{-\gamma t}\sin(t)$. We see that f satisfies conditions of Theorem 2.2. Thus, the existence of solutions follows immediately. $\square$

Chapter 3

On Sequential Caputo Boundary Value Problems with Infinite Delay

3.1 Introduction

Fractional differential equations have been able to attract a lot of attention among the current mathematical research, mainly owing to the nonlocal property of fractional derivatives. Inasmuch as they preserve and take into consideration not only the current state of the function but also past state and memory, the consideration of fractional delay differential equations is of paramount importance. In effect, there is an increasing interest in delay fractional boundary value problems; see [31–39].

Another interesting feature of fractional derivatives is their non-commutativity, which allows sequential derivatives to arise naturally, particularly in contexts where the substitution of fractional derivatives for one another is frequent. For papers including sequential boundary value problems, see [35, 39–45].

Furthermore, numerous papers have focused on fractional boundary value problems (BVP) comprising integral boundary conditions; see [42,44–51]. Namely, in the succeeding works, the authors investigated the solvability of this kind of problem. These papers are relevant to our study.

In [49], Ntouyas studied the Caputo differential equation which is subjected to three-point integral boundary conditions employing fixed point theorems:

$$
\begin{aligned}
&{}^cD^{\alpha}u(t) = f(t,u), \ t \in (0,1), \quad 1 < \alpha \le 2,\\
&u(0) = 0,\\
&u(1) = \lambda I^{\gamma}u(\eta), \ 0 < \eta < 1, \quad \lambda \neq \frac{\Gamma(\gamma+2)}{\eta^{\gamma+1}}.
\end{aligned}
$$

In [39], Li et al. proved the existence of positive solutions by using the Guo-Krasnoselskii theorem for the delay Caputo BVP with integral conditions:

$$
\begin{aligned}
&{}^cD^{\alpha}u(t) + f(t,u_t) = 0, \ t \in [0,1], \ 3 < \alpha \le 4,\\
&u(t) = \phi(t), t \in [-\tau, 0],\\
&u(0) = u''(0) = u'''(0) = 0,\\
&u(1) = \lambda \int_0^1 u(\theta)d\theta, \ 0 < \lambda < 2.
\end{aligned}
$$

In [35], Chakuvinga and Topal proved the existence of positive solutions for the p-Laplacian delayed problem stated below:

$$
\begin{aligned}
&D^{\beta}(\varphi_p({}^cD^{\alpha}u(t))) + f(t,u_t) = 0, \ t \in [0,1], \ 2 < \alpha \le 3, \ 1 < \beta \le 2,\\
&u(t) = \phi(t), t \in [-\tau, 0],\\
&u(0) = u''(0) = 0,\\
&u(1) = \lambda \int_0^1 u(\theta)d\theta, \ 0 < \lambda < 2.
\end{aligned}
$$

Inspired by the aforementioned research, we investigate the three-point sequential Caputo boundary value problem associated with infinite delay and a fractional integral boundary condition:

$$
\begin{aligned}
&{}^cD^{\alpha}({}^cD^{\beta}u)(t) = f(t,u_t), \ t \in (0,1),\\
&u(1) = \lambda I^{\gamma}u(\eta), \ \lambda > 0, \qquad (3.1)\\
&u(t) = \phi(t), \ t \in (-\infty, 0],
\end{aligned}
$$

where $0 < \alpha, \beta, \gamma, \eta < 1$, the positive constant λ is such that $\lambda \neq \frac{\Gamma(\beta+\gamma+1)}{\eta^{\beta+\gamma}\Gamma(\beta+1)}$, $f : [0,1] \times \mathbb{B} \to \mathbb{R}$ is the nonlinearity, $u_t(\theta) = u(t+\theta)$, $\theta \leq 0$, $\mathbb{B}$ is the phase space defined in the previous chapters, and the history function $\phi \in \mathbb{B}$ satisfies $\phi(0) = 0$.

It is worth noting that the research concerning sequential fractional differential equations, although addressed by numerous scholars, is still in its initial stages. There is an ongoing quest to explore different aspects of fractional sequential differential equations, many of which have not been reached yet. In a previous publication [2] we addressed the Riemann-Liouville sequential initial value problem with infinite delay. In the present chapter, a sequential delayed Caputo boundary value problem is examined, where the boundary conditions are of the fractional integral type. This work contributes to the aforementioned endeavour.

The outline of the remaining sections is given as follows: In Section 2, a subsidiary result is provided. In Section 3, we establish the solution's existence-uniqueness by virtue of the Banach contraction principle. Moreover, under weaker conditions imposed on the nonlinearity we prove the existence of solutions by using the nonlinear alternative of Leray-Schauder. In the last section, we furnish numerical examples in order to validate the findings discussed here.

3.2 Preliminaries

We furnish in this section a lemma concerning the form of the solution for the associated linear BVP.

Lemma 3.1. *Suppose that h is continuous, therefore; the solution u for the linear differential problem*

$$
\begin{aligned}
&{}^cD^{\alpha}({}^cD^{\beta}u)(t) = h(t),\ t \in (0,1),\ 0 < \alpha, \beta < 1, \\
&u(0) = 0, \\
&u(1) = \lambda I^{\gamma} u(\eta),\ 0 < \eta, \gamma < 1,
\end{aligned}
\tag{3.2}
$$

has the following form:

$$u(t) = I^{\alpha+\beta}h(t) + t^{\beta}\frac{\lambda I^{\alpha+\beta+\gamma}h(\eta) - I^{\alpha+\beta}h(1)}{\xi\Gamma(\beta+1)}$$
$$= \int_0^1 G(t,s)h(s)ds,$$

with

$$G(t,s) = \begin{cases} \frac{t^{\beta}}{\xi\Gamma(\beta+1)}(\frac{\lambda(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} - \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}) + \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}, & 0 \leq s \leq \eta, t \leq 1, \\ -\frac{t^{\beta}}{\xi\Gamma(\beta+1)}\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{t^{\beta}}{\xi\Gamma(\beta+1)}(\frac{\lambda(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} - \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}), & 0 \leq t \leq s \leq \eta \leq 1, \\ -\frac{t^{\beta}}{\xi\Gamma(\beta+1)}\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}, & 0 \leq t, \eta \leq s \leq 1, \end{cases} \tag{3.3}$$

and $\xi = \frac{1}{\Gamma(\beta+1)} - \frac{\lambda\eta^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}$. *Moreover,*

$$|\int_0^1 G(t,s)ds| \leq \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{|\xi|\Gamma(\beta+1)}(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}) := k. \tag{3.4}$$

Proof. Suppose that u is a solution for the differential equation ${}^cD^{\alpha}({}^cD^{\beta}u) = h(t)$ to which we apply the operator I^{α}. We get ${}^cD^{\beta}u(t) + c_0 = I^{\alpha}h(t)$. Now, we apply the operator I^{β} to the previous equation to obtain

$$u(t) + c_1 + \frac{c_0}{\Gamma(\beta+1)}t^{\beta} = I^{\alpha+\beta}h(t). \tag{3.5}$$

Since $u(0) = 0$, we get $c_1 = 0$. Next, we apply the operator I^{γ} to equation (3.5), we get $I^{\gamma}u(t) + \frac{c_0}{\Gamma(\beta+\gamma+1)}t^{\beta+\gamma} = I^{\alpha+\beta+\gamma}h(t)$. In view of the boundary condition $u(1) = \lambda I^{\gamma}u(\eta)$, we find $c_0 = \frac{1}{\xi}$ with $\xi = \frac{1}{\Gamma(\beta+1)} - \frac{\lambda\eta^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}$.
Consequently,

$$u(t) = \frac{t^{\beta}}{\xi\Gamma(\beta+1)}\Big(\lambda I^{\alpha+\beta+\gamma}h(\eta) - I^{\alpha+\beta}h(1)\Big) + I^{\alpha+\beta}h(t).$$

For the boundedness of G,

$$
\begin{aligned}
|\int_0^1 G(t,s)ds| \leq & \frac{t^\beta}{|\xi|\Gamma(\beta+1)}\Big|\lambda\int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}ds-\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\Big| \\
& +\Big|\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}ds\Big| \\
\leq & \frac{1}{\Gamma(\alpha+\beta+1)}+\frac{1}{|\xi|\Gamma(\beta+1)}\Big(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}+\frac{1}{\Gamma(\alpha+\beta+1)}\Big)=k.
\end{aligned}
$$

□

3.3 Existence Results

Here, we establish our existence results. First, necessary transformations are made in order to use the suitable fixed point theorems. Set $\Omega = \{u : (-\infty,1] \to \mathbb{R};\ u_{|(-\infty,0]} \in \mathbb{B}$, and $u_{|[0,1]}$ is continuous$\}$.

Definition 3.1. *We call a solution for the BVP (3.1) every function* $u \in \Omega$ *which verifies the differential equation* ${}^cD^\alpha({}^cD^\beta u(t)) = f(t,u_t)$ *on* $(0,1)$*, the boundary condition* $u(1) = \lambda I^\gamma u(\eta)$ *and* $u(t) = \phi(t)$ *on* $(-\infty,0]$.

Using Lemma 3.1, we transform the BVP (3.1) into the fixed point problem for the operator $T : \Omega \to \Omega$ where

$$
(Tu)(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ \int_0^1 G(t,s)f(s,u_s)ds, & t \in [0,1]. \end{cases} \tag{3.6}
$$

Consider the following decomposition: Define the function $x(.) : (-\infty,1] \to \mathbb{R}$ as

$$
x(t) = \begin{cases} 0, & t \in [0,1], \\ \phi(t), & t \in (-\infty,0]. \end{cases} \tag{3.7}
$$

Also, for every $v \in \mathbb{B}$ where $v_0 = 0$, $\overline{v}$ denotes the following function:

$$\overline{v}(t) = \begin{cases} v(t), & t \in [0,1], \\ 0, & t \in (-\infty, 0]. \end{cases} \tag{3.8}$$

Let u be a solution for the equation $u(t) = \int_0^1 G(t,s)f(s,u_s)ds$, $t \in [0,1]$. We partition u into $u(t) = \overline{v}(t) + x(t)$, $t \in [0,1]$. Thus $u_t = \overline{v}_t + x_t$, $t \in [0,1]$. Hence, $v(t) = \int_0^1 G(t,s)f(s,\overline{v}_s + x_s)ds$, $t \in [0,1]$.

In addition, set $C_0 = \{v \in C[0,1] : v_0 = 0\}$ endowed with the semi-norm in C_0 given as $\|v\|_1 = \|v_0\|_{\mathbb{B}} + \sup\limits_{t\in[0,1]} |v(t)| = \sup\limits_{t\in[0,1]} |v(t)|$. Consider now the operator $\mathscr{N} : C_0 \to C_0$ given by

$$(\mathscr{N} v)(t) = \int_0^1 G(t,s)f(s,\overline{v}_s + x_s)ds. \tag{3.9}$$

Claiming that T possesses a fixed point is the same as claiming that the operator $\mathscr{N}$ has one too. Firstly, we give the following existence-uniqueness result, which is due to the Banach contraction principle.

Theorem 3.1. *Let f be continuous and fulfills the Lipschitz condition:*

$(H1)$ *There exists a nonnegative $L > 0$ verifying*

$$|f(t,u) - f(t,v)| \leq L\|u - v\|_{\mathbb{B}}, \text{ for every } u, v \in \mathbb{B}$$

where $LK_1k < 1$, k is given in (3.4), *and $K_1 = \sup\limits_{t\in[0,1]} K(t)$. Then there exists a unique solution for the BVP (3.1).*

Proof. Indeed, $\mathscr{N}$ is a contraction. Take $v, v^* \in C_0$. Hence, using (1.9), for every $t \in [0,1]$,

$$\begin{aligned} |\mathscr{N} v(t) - \mathscr{N} v^*(t)| &\leq \int_0^1 |G(t,s)||f(s,\overline{v}_s + x_s) - f(s,\overline{v_s^*} + x_s)|ds \\ &\leq \int_0^1 |G(t,s)|L\|\overline{v}_s - \overline{v_s^*}\|_{\mathbb{B}} ds \end{aligned}$$

$$
\begin{aligned}
&\leq L\int_0^1 |G(t,s)|K(s)\sup_{\theta\in[0,s]}|v(\theta)-v^*(\theta)|ds \\
&\leq LK_1\|v-v^*\|_1\int_0^1 |G(t,s)|ds \\
&\leq LK_1 k\|v-v^*\|_1.
\end{aligned}
$$

Hence, $\mathscr{N}$ is a contraction mapping. Consequently, employing Banach's contraction principle, we infer that it admits a unique fixed point. □

Next, we provide the subsequent existence result.

Theorem 3.2. *Suppose that f is continuous and verifying the Lipschitz condition: There exists $L>0$, $|f(t,u)-f(t,v)|\leq L\|u-v\|_{\mathbb{B}}$, for every $t\in[0,1]$, $u\in\mathbb{B}$ such that*

$$\frac{LK_1}{|\xi|\Gamma(\beta+1)}\left(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}+\frac{1}{\Gamma(\alpha+\beta+1)}\right)<1. \tag{3.10}$$

Moreover, assume that there exists $\mu\in C([0,1],\mathbb{R}_+)$ with

$$|f(t,u)|\leq\mu(t),\ t\in[0,1],\ u\in\mathbb{B}. \tag{3.11}$$

Thus, the BVP (3.1) possesses a solution on $[0,1]$.

Proof. The proof is due to the Krasnoselskii's fixed point theorem. We give the decomposition: $\mathscr{N}=\mathscr{N}_1+\mathscr{N}_2$, where the operator $\mathscr{N}_1: C_0\to C_0$ is defined as

$$\mathscr{N}_1 v(t)=\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,\overline{v}_s+x_s)ds, \tag{3.12}$$

and the operator $\mathscr{N}_2: C_0\to C_0$ is given by

$$
\begin{aligned}
\mathscr{N}_2 v(t)=\frac{t^\beta}{\xi\Gamma(\beta+1)}\Big(\lambda\int_0^\eta &\frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}f(s,\bar{v_s}+x_s)ds \\
&-\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,\bar{v_s}+x_s)ds\Big).
\end{aligned} \tag{3.13}
$$

Choose

$$R \geq \frac{\|\mu\|}{\Gamma(\alpha+\beta+1)} + \frac{\|\mu\|}{\xi\Gamma(\beta+1)}\Big(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}\Big)$$

and $B_R \subset C_0$ given by $B_R = \{v \in C_0 : \|v\|_1 \leq R\}$. Set $\mathcal{N}_1 : B_R \to C_0$ and $\mathcal{N}_2 : B_R \to C_0$. We will prove that for every $v_1, v_2 \in B_R$ we have $\mathcal{N}_1 v_1 + \mathcal{N}_2 v_2 \in B_R$, the operator $\mathcal{N}_1$ is continuous and compact, and the operator $\mathcal{N}_2$ is a contraction mapping. Indeed, for every $t \in [0,1]$,

$$\begin{aligned}
|\mathcal{N}_1 v_1(t) + \mathcal{N}_2 v_2(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,\overline{v_1}_s + x_s)|ds + \frac{1}{|\xi|\Gamma(\beta+1)}\times \\
&\Big(\lambda\int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}|f(s,\overline{v_2}_s + x_s)|ds + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,\overline{v_2}_s + x_s)|ds\Big) \\
&\leq \|\mu\|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds \\
&+ \frac{\|\mu\|}{|\xi|\Gamma(\beta+1)}\Big(\lambda\int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}ds + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\Big) \\
&\leq \frac{\|\mu\|}{\Gamma(\alpha+\beta+1)} + \frac{\|\mu\|}{|\xi|\Gamma(\beta+1)}\Big(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}\Big) \leq R.
\end{aligned}$$

After that, we prove that $\mathcal{N}_2$ is a contraction. Let $v, v^* \in B_R$. Hence, for all $t \in [0,1]$,

$$\begin{aligned}
|\mathcal{N}_2 v(t) - \mathcal{N}_2 v^*(t)| \leq &\frac{t^\beta}{|\xi|\Gamma(\beta+1)}\Big(\lambda\int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}|f(s,\bar{v}_s + x_s) - f(s,\overline{v_s^*} + x_s)|ds \\
&+ \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,\bar{v}_s + x_s) - f(s,\overline{v_s^*} + x_s)|ds\Big) \\
&\leq \frac{L}{|\xi|\Gamma(\beta+1)}\Big(\lambda\int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}\|\bar{v}_s - \overline{v_s^*}\|_{\mathbb{B}}ds \\
&+ \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|\bar{v}_s - \overline{v_s^*}\|_{\mathbb{B}}ds\Big) \\
&\leq \frac{L}{|\xi|\Gamma(\beta+1)}\Big(\lambda\int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)}K(s)\sup_{\theta\in[0,s]}|v(\theta) - v^*(\theta)|ds \\
&+ \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}K(s)\sup_{\theta\in[0,s]}|v(\theta) - v^*(\theta)|ds\Big) \\
&\leq \frac{LK_1\|v - v^*\|_1}{|\xi|\Gamma(\beta+1)}\Big(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}\Big) < \|v - v^*\|_1.
\end{aligned}$$

Next, we prove that $\mathscr{N}_1$ is continuous and completely continuous.
First, since f is continuous, $\mathscr{N}_1$ is a continuous operator. Second, $\mathscr{N}_1$ is uniformly bounded on B_R : $|\mathscr{N}_1 v(t)| \leq \frac{\|\mu\|_\infty}{\Gamma(\alpha+\beta+1)}$. Now, we prove the equicontinuity of the operator $\mathscr{N}_1$. Take $0 \leq t_1 < t_2 \leq 1$, and let $v \in B_R$. Then

$$\begin{aligned}
|(\mathscr{N}_1 v)(t_2) - (\mathscr{N}_1 v)(t_1)| \leq & \frac{1}{\Gamma(\alpha+\beta)}\Big(\int_0^{t_1} |(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}||f(s,\overline{v}_s+x_s)|ds \\
& + \int_{t_1}^{t_2} |(t_2-s)^{\alpha+\beta-1} f(s,\overline{v}_s+x_s)|ds\Big) \\
\leq & \frac{\|\mu\|_\infty}{\Gamma(\alpha+\beta)} \int_0^{t_1} |(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}|ds \\
& + \frac{\|\mu\|_\infty}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} ds,
\end{aligned}$$

which approaches zero, as t_1 tends to t_2. Hence, operator $\mathscr{N}_1$ is indeed equicontinuous. By virtue of the Arzelà-Ascoli theorem, we infer that $\mathscr{N}_1$ is compact. Therefore, employing the Krasnoselskii's fixed point theorem, the operator $\mathscr{N} = \mathscr{N}_1 + \mathscr{N}_2$ admits a fixed point in B_R, which is also a solution of the BVP (3.1). □

Finally, under a growth condition assumed on the nonlinearity, we give another existence result.

Theorem 3.3. *Suppose that*

1. *f is continuous,*

2. *there exist $p, q \in C([0,1], \mathbb{R}_+)$ verifying*

$$\begin{aligned}
&|f(t,u)| \leq p(t) + q(t)\|u\|_{\mathbb{B}}, \quad t \in [0,1],\ u \in \mathbb{B}, \\
&\|q\|_\infty < \frac{1}{kK_1},
\end{aligned}$$

then the BVP (3.1) *admits a solution on* $[0,1]$.

Proof. Let $\mathscr{N} : C_0 \to C_0$ be the operator given in (3.9). It is clear that $\mathscr{N}$ is a continuous operator. We shall prove that $\mathscr{N}$ is completely continuous.

First, $\mathcal{N}$ is uniformly bounded in C_0.

Define $B_R := \{v \in C_0 : \|v\|_1 \leq R\}$. Take $v \in B_R$, $t \in [0,1]$, therefore;

$$\begin{aligned}
|\mathcal{N} v(t)| \leq & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s,\bar{v}_s + x_s)| ds + \frac{1}{|\xi|\Gamma(\beta+1)} \times \\
& \Big(\lambda \int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} |f(s,\bar{v}_s + x_s)| ds + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,\bar{v}_s + x_s)| ds\Big) \\
& \leq \|p\|_\infty \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds + \frac{\|p\|_\infty}{|\xi|\Gamma(\beta+1)} \times \Big(\lambda \int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds\Big) + \|q\|_\infty \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \|\bar{v}_s + x_s\|_{\mathbb{B}} ds + \frac{\|q\|_\infty}{|\xi|\Gamma(\beta+1)} \times \\
& \Big(\lambda \int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} \|\bar{v}_s + x_s\|_{\mathbb{B}} ds + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \|\bar{v}_s + x_s\|_{\mathbb{B}} ds\Big) \\
& \leq (k\|p\|_\infty + R^* k \|q\|_\infty) := l,
\end{aligned}$$

where $\|\bar{v}_s + x_s\|_{\mathbb{B}} \leq \|\bar{v}_s\|_{\mathbb{B}} + \|x_s\|_{\mathbb{B}} \leq K_1 R + M_1 \|\phi\|_{\mathbb{B}} := R^*$, and $M_1 = \sup_{t\in[0,1]} M(t)$. Hence, $\|\mathcal{N} v\|_1 \leq l$.

Second, we show that $\mathcal{N}$ is uniformly equicontinuous. Take $0 \leq t_1 < t_2 \leq 1$ and B_R defined as above. Let $v \in B_R$,

$$\begin{aligned}
|(\mathcal{N} v)(t_2) - (\mathcal{N} v)(t_1)| \leq & \frac{1}{\Gamma(\alpha+\beta)} \Big(\int_0^{t_1} |(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}| |f(s,\bar{v}_s + x_s)| ds \\
& + \int_{t_1}^{t_2} |(t_2-s)^{\alpha+\beta-1} f(s,\bar{v}_s + x_s)| ds\Big) + |t_2^\beta - t_1^\beta| \Big(\lambda \int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} f(s,\bar{v}_s + x_s) ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,\bar{v}_s + x_s) ds\Big) \\
& \leq \frac{\|p\|_\infty + \|q\|_\infty R^*}{\Gamma(\alpha+\beta)} \Big(\int_0^{t_1} |(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}| ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} ds\Big) \\
& + |t_2^\beta - t_1^\beta| \frac{(\|p\|_\infty + R^* \|q\|_\infty)}{|\xi|\Gamma(\beta+1)} \Big(\frac{\lambda \eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}\Big),
\end{aligned}$$

which approaches zero, as t_1 tends to t_2. Owing to the previous steps, besides the Arzelà-Ascoli theorem, $\mathcal{N} : C_0 \to C_0$ is completely continuous.

Now, it remains to prove the existence of an open set $U \subseteq C_0$ such that $v \neq \nu \mathcal{N} v$, $\nu \in (0,1)$,

$v \in \partial U$. Take $v \in C_0$ and $v = \nu \mathscr{N}(v)$, $0 < \nu < 1$. Then for all $t \in [0,1]$,

$$\begin{aligned}
|v(t)| \leq & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} (p(s) + q(s) \|\overline{v}_s + x_s\|_{\mathbb{B}}) ds \\
& + \frac{1}{|\xi|\Gamma(\beta+1)} \Big(\lambda \int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} (p(s) + q(s) \|\overline{v}_s + x_s\|_{\mathbb{B}}) ds + \\
& \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} (p(s) + q(s) \|\overline{v}_s + x_s\|_{\mathbb{B}}) ds \Big) \\
& \leq k\|p\|_\infty + \|q\|_\infty \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \|\overline{v}_s + x_s\|_{\mathbb{B}} ds + \frac{\|q\|_\infty}{|\xi|\Gamma(\beta+1)} \times \\
& \Big(\lambda \int_0^\eta \frac{(\eta-s)^{\alpha+\beta+\gamma-1}}{\Gamma(\alpha+\beta+\gamma)} \|\overline{v}_s + x_s\|_{\mathbb{B}} ds + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \|\overline{v}_s + x_s\|_{\mathbb{B}} ds \Big).
\end{aligned}$$

Then, by virtue of (1.9), $\|\overline{v}_s + x_s\|_{\mathbb{B}} \leq K(s) \sup_{\theta \in [0,s]} |v(\theta)| + M(s)\|\phi\|_{\mathbb{B}} \leq K_1\|v\|_1 + M_1\|\phi\|_{\mathbb{B}}$, which yields

$$|v(t)| \leq k\|p\|_\infty + k\|q\|_\infty (K_1\|v\|_1 + M_1\|\phi\|_{\mathbb{B}}),\ t \in [0,1].$$

Then $\|v\|_1 \leq \dfrac{k\|p\|_\infty + k\|q\|_\infty M_1\|\phi\|_{\mathbb{B}}}{1 - k\|q\|_\infty K_1} := r^*$.

Set $U = \{v \in C_0 : \|v\|_1 < r^* + 1\}$. Then $\mathscr{N} : \overline{U} \to C_0$ is continuous and completely continuous. Based on the choice of U, there is no $v \in \partial U$ satisfying $v = \nu \mathscr{N}(v)$, for $\nu \in (0,1)$. Hence, the nonlinear alternative of Leray-Schauder is applicable. Thus, $\mathscr{N}$ possesses a fixed point v in U. □

3.4 Examples

We validate our findings through numerical examples.

Example 3.1. *Let* $\alpha = \beta = \eta = \gamma = \frac{1}{2}, \lambda = 1$, *and* $\tau > 0$. *Then*

$$\begin{aligned}
&\Gamma(\alpha+\beta+1) = \Gamma(\beta+\gamma+1) = \Gamma(2) = 1, \\
&\Gamma(\beta+1) = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}, \\
&\Gamma(\alpha+\beta+\gamma+1) = \Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}.
\end{aligned}$$

Thus, $\dfrac{\Gamma(\beta+\gamma+1)}{\eta^{\beta+\gamma}\Gamma(\beta+1)} = \dfrac{4}{\sqrt{\pi}} \neq \lambda$. *Hence,*

$$\begin{aligned} k =& \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{|\xi|\Gamma(\beta+1)}\left(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}\right) \\ =& 1 + \frac{4}{4-\sqrt{\pi}}\left(\frac{2}{3\sqrt{2\pi}} + 1\right) = 3,2734. \end{aligned}$$

Now, we choose $f : [0,1] \times \mathbb{B}_\tau \to \mathbb{R}$ *given by*

$$f(t,u) = \frac{3e^{-\tau t}}{10}\left(\frac{u}{1+u}\right),$$

and $\phi \in \mathbb{B}_\tau$ *which is the space defined by*

$$\mathbb{B}_\tau = \{\phi : C((-\infty,0],\mathbb{R}) : \lim_{s\to-\infty} e^{\tau s}|\phi(s)| \textit{exists in } \mathbb{R}\}$$

and endowed with the norm $\|\phi\|_{\mathbb{B}_\tau} = \sup_{s\in(-\infty,0]} e^{\tau s}|\phi(s)|$. *It is easy to see that* $\mathbb{B}_\tau$ *is an admissible phase space,* ***i.e.***, *it is a Banach space and it fulfills the phase space axioms with* $K(t) = 1$, $M(t) = e^{-\tau t}$ *and* $H = 1$.
Moreover, for all $t \in [0,1]$, $u, v \in \mathbb{B}_\tau$

$$\begin{aligned} |f(t,u) - f(t,v)| &= \frac{3e^{-\tau t}}{10}\left|\frac{u}{1+u} - \frac{v}{1+v}\right| \\ &\leq \frac{3e^{-\tau t}}{10}\frac{|u-v|}{(1+u)(1+v)} \\ &\leq \frac{3e^{-\tau t}}{10}|u-v| \leq \frac{3}{10}\|u-v\|_{\mathbb{B}_\tau}, \end{aligned}$$

so f *verifies the condition* $(H1)$ *with* $L = \frac{3}{10}$ *and* $LK_1k = 0,982 < 1$, *therefore, the BVP* (3.1) *has exactly one solution by virtue of Theorem 3.1. Additionally,*

$$\begin{aligned} |f(t,u)| &\leq \frac{3e^{-\tau t}}{10}\left|\frac{u}{1+u}\right| \\ &\leq \frac{3}{10}e^{-\tau t}|u(t)| \end{aligned}$$

$$\leq \frac{3}{10}\|u\|_{\mathbb{B}_\tau} = p(t) + q(t)\|u\|_{\mathbb{B}_\tau},$$

with $p(t) = 0$, *and* $q(t) = \frac{3}{10}$, *then* $\|q\|_\infty < \frac{1}{kK_1}$. *Thus, the conditions of Theorem 3.3 are also verified. Hence, the existence of solutions follows immediately.*

Example 3.2. *Take* $\alpha = \frac{2}{5}, \beta = \frac{1}{5}, \eta = \frac{1}{2}, \gamma = 1, \lambda = 1$, *and* $\tau > 0$. *Then*

$$\Gamma(\alpha+\beta+1) = \Gamma(\frac{8}{5}) = 0,8935,$$
$$\Gamma(\beta+\gamma+1) = \Gamma(\frac{11}{5}) = 1,1018,$$
$$\Gamma(\beta+1) = \Gamma(\frac{6}{5}) = 0,9182,$$
$$\Gamma(\alpha+\beta+\gamma+1) = \Gamma(\frac{13}{5}) = 1,4296.$$

Then $\dfrac{\Gamma(\beta+\gamma+1)}{\eta^{\beta+\gamma}\Gamma(\beta+1)} = 2,7568 \neq \lambda$. *Moreover,*

$$\frac{1}{|\xi|\Gamma(\beta+1)}(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}) = 2,1186.$$

The nonlinearity $f : [0,1] \times \mathbb{B}_\tau \to \mathbb{R}$, *with the phase space* $\mathbb{B}_\tau$ *defined as above, is given by*

$$f(t,u) = \frac{e^{-\tau t}}{(t+12)^2}(\frac{64}{1+|u|}).$$

Furthermore, for all $t \in [0,1]$, $u, v \in \mathbb{B}_\tau$,

$$\begin{aligned} |f(t,u) - f(t,v)| &= \frac{64e^{-\tau t}}{(t+12)^2}|\frac{1}{1+|u|} - \frac{1}{1+|v|}| \\ &\leq \frac{64e^{-\tau t}}{(t+12)^2}\frac{\Big||u|-|v|\Big|}{(1+|u|)(1+|v|)} \\ &\leq \frac{64}{144}e^{-\tau t}|u-v| \\ &\leq \frac{4}{9}\|u-v\|_{\mathbb{B}_\tau}. \end{aligned}$$

Since $k = 1,1192 + 2,1186 = 3,2378$, *then* $LK_1k = 1,439 > 1$. *In this case, Theorem 3.1* *does not apply. However,* $\frac{LK_1}{|\xi|\Gamma(\beta+1)}(\frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}+\frac{1}{\Gamma(\alpha+\beta+1)})=0,9416<1$. *Moreover, for every* $t\in[0,1]$, $u\in\mathbb{B}_\tau$,

$$|f(t,u)|\leq\frac{64e^{-\tau t}}{(t+12)^2}\frac{1}{1+|u|}\leq\frac{64e^{-\tau t}}{(t+12)^2}:=\mu(t).$$

Consequently, f *verifies the conditions of Theorem* 3.2*. Thus, the existence of solutions follows immediately.*

Chapter 4

On Sequential Caputo Boundary Value Problems at Resonance

4.1 Introduction

In this chapter, we examine, via Mawhin's coincidence degree theory, a three-point sequential Caputo boundary value problem at resonance, subject to fractional integral boundary conditions:

$$\begin{aligned} &{}^{c}D^{\alpha c}D^{\beta}u(t) = f(t,u), t \in (0,1], \\ &u(1) = \lambda I^{\gamma}u(\eta), \\ &u(0) = 0, \end{aligned} \tag{4.1}$$

where $0 < \alpha, \beta, \gamma, \eta < 1$, $\lambda = \frac{\Gamma(\beta+\gamma+1)}{\eta^{\beta+\gamma}\Gamma(\beta+1)}$, and $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is the nonlinearity.

Mawhin introduced coincidence degree theory to study both functional and differential equations [13]. Later, this theory is expanded, making such important contributions that it became known as Mawhin's coincidence degree theory. This theory is very useful in solving problems involving nonlinear differential equations.

The main objective of coincidence degree theory is to inspect solutions for an operator equation $Lx = Nx$ within a specific set Ω in a Banach space. This theory uses the Leray-Schauder degree theory, focusing on a linear operator L and a nonlinear operator N.

In finite dimensions, a well-defined degree exists for certain functions within specific sets. However, in infinite dimensions, this is not always the case. Leray and Schauder showed that, in an arbitrary Banach space, a well-defined degree exists for specific compact operators within certain bounded sets.

The significance of degree theory lies in determining whether a certain equation has solutions within a set. Mawhin's work focused on investigating solutions for $Lx = Nx$ within a Banach space using Leray-Schauder degree theory. As the operator $I - (L - N)$ is not typically compact, a compact operator M is introduced to align the fixed points of M within Ω with the solutions of $Lx = Nx$ within the same set, and then the coincidence degree for the pair (L,N) is defined in Ω as $\deg[(L,N),\Omega] = \deg(I - M,\Omega,0)$.

Many scholars have focused on boundary value problems (BVPs) at resonance, employing Mawhin's degree theory, across various papers. Nieto, in [52], studied nonlocal second order boundary value problems at resonance by applying Mawhin's coincidence degree theory. Specifically, he considered the following problem:

$$\begin{aligned} &u''(t) = f(t,u), t \in [0,b], \\ &u(0) = 0, \quad \alpha u(\eta) = u(b), \end{aligned}$$

with $b > 0$, $\eta \in (0,b)$, $\alpha\eta = b$, and $f\colon [0,b] \times \mathbb{R} \to \mathbb{R}$ being continuous.

Furthermore, there has been a growing interest in dealing with fractional boundary value problems (BVPs). These problems serve as valuable mathematical tools for describing various physical phenomena, resulting in an increasing number of papers focusing on the existence, uniqueness, positivity, and stability of their solutions. Several functional analysis tools have been dedicated to their examination, namely Leray-Schauder's fixed point theorem, the Banach contraction principle, and Krasnoselskii's fixed point theorem. However, these theorems do not apply when the homogeneous problem has a nontrivial solution, which characterises a BVP at resonance. For more results on BVPs at resonance, see [53–61], [62–68] for three-point BVPs, and [69–73] for BVPs with integral conditions.

The outline for the remaining sections of this chapter is given: Section 2 covers the necessary lemmas, serving as a basis for the proof of our main result, and it provides the main existence result due to Mawhin's coincidence degree theory. In Section 3, we validate the applicability of our findings via a numerical example.

4.2 Main Results

Let us start by introducing the succeeding results, which will be utilized to prove the main theorem.

Define the operator $L\colon X \to Y$ as $Lx(t) = ^c D^{\alpha c}D^{\beta}x(t)$ and define $N\colon X \to Y$ as $Nx(t) = f(t,x(t))$ with $X = C[0,1]$ and $Y = C[0,1]$ are endowed with the norm $\|x\|_X = \|x\|_Y = \max_{t\in[0,1]} |x(t)| := \|x\|$ and

$$domL = \{x \in X,\, x(0) = 0,\, x(1) = \lambda I^{\gamma} x(\eta)\}.$$

Thus,

$$KerL = \{a\frac{t^{\beta}}{\Gamma(\beta+1)}, a \in \mathbb{R}\},$$

$$ImL = \{y \in Y, I^{\alpha+\beta}y(1) = \lambda I^{\alpha+\beta+\gamma}y(\eta)\}.$$

Lemma 4.1. *Operator L is a Fredholm Operator of index zero.*

Proof. Let $Q\colon Y \to Y$ be the mapping defined by $Qy = A^{-1}(I^{\alpha+\beta}y(1) - \lambda I^{\alpha+\beta+\gamma}y(\eta))$, where $A = \frac{1}{\Gamma(\alpha+\beta+1)} - \frac{\eta^{\alpha}\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)\Gamma(\beta+1)}$. Observe that $KerQ = ImL$.

We claim that the linear continuous mapping Q is a projection, i.e., it satisfies $Q^2y = Qy$. Indeed,

$$\begin{aligned}
Q(Qy) &= A^{-1}((I^{\alpha+\beta}Qy)(1) - \lambda(I^{\alpha+\beta+\gamma}Qy)(\eta)) \\
&= A^{-1}\Big(\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}A^{-1}(I^{\alpha+\beta}y(1) - \lambda I^{\alpha+\beta+\gamma}y(\eta))ds \\
&- \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+\gamma-1}A^{-1}(I^{\alpha+\beta}y(1) - \lambda I^{\alpha+\beta+\gamma}y(\eta))ds\Big) \\
&= A^{-2}(I^{\alpha+\beta}y(1) - \lambda I^{\alpha+\beta+\gamma}y(\eta))\Big(\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}ds \\
&- \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+\gamma-1}ds\Big) \\
&= A^{-2}(I^{\alpha+\beta}y(1) - \lambda I^{\alpha+\beta+\gamma}y(\eta))\Big(\frac{1}{\Gamma(\alpha+\beta+1)} - \frac{\lambda\eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}\Big) \\
&= A^{-2}(I^{\alpha+\beta}y(1) - \lambda I^{\alpha+\beta+\gamma}y(\eta))A = Qy.
\end{aligned}$$

Let $y \in Y$, then $y = (y - Qy) + Qy$. Since $(y - Qy) \in KerQ = ImL$, $Y \subset ImL + ImQ$. Furthermore, we easily check that $ImL \cap ImQ = \{0\}$. Thus, $Y = ImL \oplus ImQ$.

Define $P\colon X \to X$ by $Px(t) = {}^c D^\beta x(0)\frac{t^\beta}{\Gamma(\beta+1)}$. Notice that $ImP = KerL$ and $P^2x = Px$ since ${}^cD^\beta(t^\beta) = \Gamma(\beta+1)$. In fact,

$$P^2x(t) = P(Px)(t) = ({}^cD^\beta Px)(0)\frac{t^\beta}{\Gamma(\beta+1)} = {}^c D^\beta x(0)\frac{t^\beta}{\Gamma(\beta+1)} = Px(t).$$

We have $KerP = \{x \in X, {}^c D^\beta x(0) = 0\}$.

Let $x \in X$, thus $x = (x - Px) + Px$. Since $(x - Px) \in KerP$ and $Px \in ImP = KerL$, we have $X \subset KerL + KerP$. We verify that $KerP \cap KerL$=$\{0\}$; hence, $X = KerL \oplus KerP$.

Define $K_P\colon ImL \to domL \cap KerP$ by $K_Py = I^{\alpha+\beta}y$. The operator K_P makes sense, i.e., for every $y \in ImL$, $K_Py \in domL \cap KerP$. Indeed, for all $y \in ImL$, $K_Py(0) = 0$ and $K_Py(1) = \lambda I^\gamma K_Py(\eta)$.

$$LK_Py = {}^c D^{\beta\,c}D^\alpha I^{\alpha+\beta}y = {}^c D^{\beta\,c}D^\alpha I^\alpha(I^\beta y) = {}^c D^\beta I^\beta y = y, \text{ for every } y \in ImL.$$

For $x \in domL \cap KerP$, since $x(0) = 0, {}^c D^\beta x(0) = 0$, we get

$$K_PLx = I^{\alpha+\beta}({}^cD^\alpha({}^cD^\beta x)) = I^\beta(I^{\alpha\,c}D^\alpha({}^cD^\beta x)) = I^{\beta\,c}D^\beta x(t) = I^{\beta\,c}D^\beta x(t) = x(t).$$

□

Hereinafter, we consider the assumptions given below:

(H1) There exist two functions $p, q \in C([0,1],[0,+\infty))$ satisfying $1 - R\|q\| > 0$,

$$|f(t,x(t))| \leq p(t) + q(t)|x(t)|, \text{ for every } t \in [0,1],\ x \in X,$$

where $R = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+1)}$.

(H2) There exists $k > 0$, for every $x \in domL$ such that $|{}^cD^\beta x(t)| > k$, for every $t \in [0,1]$ implies $I^{\alpha+\beta}Nx(1) - \lambda I^{\alpha+\beta+\gamma}Nx(\eta) \neq 0$.

(H3) There exists $K > 0$ such that for $|a| > K$, either

$$a\Big(I^{\alpha+\beta} f(s, a\frac{s^\beta}{\Gamma(\beta+1)})(1) - \lambda I^{\alpha+\beta+\gamma} f(s, a\frac{s^\beta}{\Gamma(\beta+1)})(\eta)\Big) > 0$$

or

$$a\Big(I^{\alpha+\beta} f(s, a\frac{s^\beta}{\Gamma(\beta+1)})(1) - \lambda I^{\alpha+\beta+\gamma} f(s, a\frac{s^\beta}{\Gamma(\beta+1)})(\eta)\Big) < 0.$$

Lemma 4.2. *Assume (H1) holds, then the operator N is $L-$compact on $\overline{\mathscr{U}}$ for any open bounded set $\mathscr{U} \subset X$.*

Proof. Let $\mathscr{U} = \{x \in X, \|x\|_X < r\}$. First, we easily verify that QN is a continuous operator. Second, we prove that $QN(\overline{\mathscr{U}})$ is bounded.
Let $y \in \overline{\mathscr{U}}$, then

$$\begin{aligned}
|QNy(t)| \le |A^{-1}|\Big(&\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|f(s,y(s))|ds \\
&+ \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)}\int_0^\eta (\eta-s)^{\alpha+\beta+\gamma-1}|f(s,y(s))|ds\Big) \\
&\le |A^{-1}|\Big(\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(p(s)+q(s)|y(s)|)ds \\
&+ \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)}\int_0^\eta (\eta-s)^{\alpha+\beta+\gamma-1}(p(s)+q(s)|y(s)|)ds\Big) \\
&\le |A^{-1}|(\|I^{\alpha+\beta}p\|_\infty + \lambda\|I^{\alpha+\beta+\gamma}p\|_\infty) + r|A^{-1}|(\|I^{\alpha+\beta}q\|_\infty + \lambda\|I^{\alpha+\beta+\gamma}q\|_\infty).
\end{aligned}$$

Thus,

$$\|QNy\|_Y \le |A^{-1}|\Big((\|I^{\alpha+\beta}p\|_\infty + \lambda\|I^{\alpha+\beta+\gamma}p\|_\infty) + r(\|I^{\alpha+\beta}q\|_\infty + \lambda\|I^{\alpha+\beta+\gamma}q\|_\infty)\Big).$$

Now, we prove that $K_P(I-Q)N\colon \overline{\mathscr{U}} \to X$ is completely continuous. Employing the Arzelà-Ascoli theorem, it is sufficient to prove that $K_P(I-Q)N\colon \overline{\mathscr{U}} \to X$ is equicontinuous and bounded.

First, for every $y \in \overline{\mathscr{U}}$, $t \in [0,1]$,

$$\begin{aligned} K_P(I-Q)Ny(t) &= K_P(Ny - QNy)(t) = I^{\alpha+\beta}(Ny - QNy)(t) \\ &= (I^{\alpha+\beta}Ny)(t) - (I^{\alpha+\beta}QNy)(t) \\ &= (I^{\alpha+\beta}Ny)(t) - \frac{A^{-1}}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}(I^{\alpha+\beta}Ny(1) - \lambda I^{\alpha+\beta+\gamma}Ny(\eta)). \end{aligned}$$

Therefore,

$$|K_P(I-Q)Ny(t)| \le \frac{\|p\| + \|q\|r}{\Gamma(\alpha+\beta+1)}(1 + |A^{-1}|(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\lambda \eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)})) := R_1. \tag{4.2}$$

We obtain

$$\|K_P(I-Q)Ny\|_Y \le R_1.$$

Second, $K_P(I-Q)N(\overline{\mathscr{U}})$ is equicontinuous. Let $0 < t_1 < t_2 < 1$ and $y \in \overline{\mathscr{U}}$. Thus,

$$\begin{aligned} &|K_P(I-Q)Ny(t_2) - K_P(I-Q)Ny(t_1)| \le |I^{\alpha+\beta}Ny(t_2) - I^{\alpha+\beta}Ny(t_1)| \\ &+ \frac{|A^{-1}|}{\Gamma(\alpha+\beta+1)}|t_2^{\alpha+\beta} - t_1^{\alpha+\beta}||I^{\alpha+\beta}Ny(1) - \lambda I^{\alpha+\beta+\gamma}Ny(\eta)|. \\ &\le \frac{\|p\| + \|q\|r}{\Gamma(\alpha+\beta+1)}(\int_0^{t_1} |(t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1}|ds + \int_{t_1}^{t_2} |(t_2 - s)^{\alpha+\beta-1}|ds) \\ &+ \frac{\|p\| + \|q\|r}{\Gamma(\alpha+\beta+1)}(|A^{-1}||t_2^{\alpha+\beta} - t_1^{\alpha+\beta}|(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\lambda \eta^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)})) \end{aligned}$$

which approaches 0, when $t_1 \to t_2$. □

Lemma 4.3. *Suppose that $(H1)$ and $(H2)$ hold. Therefore, the set*

$$\{x \in domL \setminus KerL,\ Lx = \mu Nx,\ \mu \in (0,1)\}$$

is bounded.

Proof. Since $Nx \in ImL$, $I^{\alpha+\beta}Nx(1) - \lambda I^{\alpha+\beta+\gamma}Nx(\eta) = 0$. Using the assumption (H2), there exists $t_0 \in [0,1]$ such that $|^cD^\beta x(t_0)| \leq k$. Thus, for all $t \in [0,1]$,

$$|^cD^\beta x(0)| = |^cD^\beta x(t) - I^\alpha(^cD^{\alpha c}D^\beta x)(t)|.$$

Hence, in particular,

$$|^cD^\beta x(0)| \leq |^cD^\beta x(t_0)| + |I^\alpha Lx(t_0)| \leq k + \frac{\|Nx\|}{\Gamma(\alpha+1)}.$$

On the other hand, $x \in domL \setminus KerL$, so $(I-P)x \in KerP \cap domL$ and $Px \in KerL$. We have

$$\begin{aligned}
\|x\| &\leq \|Px\| + \|(I-P)x\| \\
&\leq \frac{|^cD^\beta x(0)|}{\Gamma(\beta+1)} + \|K_P L(I-P)x\| \\
&\leq \frac{1}{\Gamma(\beta+1)}(k + \frac{\|Lx\|}{\Gamma(\alpha+1)}) + \frac{1}{\Gamma(\alpha+\beta+1)}\|Lx\| \\
&\leq \frac{k}{\Gamma(\beta+1)} + R\|Nx\| \leq \frac{k}{\Gamma(\beta+1)} + R(\|p\| + \|x\|\|q\|).
\end{aligned}$$

Since $1 - R\|q\| > 0$,

$$\|x\| \leq \frac{1}{1-R\|q\|}(\frac{k}{\Gamma(\beta+1)} + R\|p\|) := T < \infty.$$

□

Lemma 4.4. *Assume (H2) holds. Then the set*

$$\{x \in KerL, \, Nx \in ImL\}$$

is bounded.

Proof. Let $x \in KerL$ such that $Nx \in ImL$, then $x(t) = a\frac{t^\beta}{\Gamma(\beta+1)}$ and

$$I^{\alpha+\beta}Nx(1) - \lambda I^{\alpha+\beta+\gamma}Nx(\eta) = 0.$$

By hypothesis (H2), we conclude the existence of $t_1 \in (0,1)$ such that $|{}^cD^\beta x(t_1)| = |a| \le k$, so $|x(t)| \le \frac{k}{\Gamma(\beta+1)}$. Thus, $\|x\| \le \frac{k}{\Gamma(\beta+1)} := M$. Hence, the set is bounded. □

Lemma 4.5. *Assume (H2) and (H3) hold true. Either the set*

$$\{x \in KerL,\ \mu x + (1-\mu)QNx = 0,\ \mu \in [0,1]\}$$

or the set

$$\{x \in KerL,\ -\mu x + (1-\mu)QNx = 0,\ \mu \in [0,1]\}$$

is bounded.

Proof. Take $x \in KerL$ such that $\mu x + (1-\mu)QNx = 0$. Then $x(t) = a\frac{t^\beta}{\Gamma(\beta+1)}$ and $\mu a\frac{t^\beta}{\Gamma(\beta+1)} + (1-\mu)QNx = 0$, i.e.,

$$\mu a\frac{t^\beta}{\Gamma(\beta+1)} + (1-\mu)(I^{\alpha+\beta}Nx(1) - \lambda I^{\alpha+\beta+\gamma}Nx(\eta)) = 0. \tag{4.3}$$

If $\mu = 0$, then $I^{\alpha+\beta}Nx(1) - \lambda I^{\alpha+\beta+\gamma}Nx(\eta) = 0$, and following the same argument as in Lemma 4.4, we get $|x(t)| \le \frac{k}{\Gamma(\beta+1)}$.

If $\mu = 1$, then $a = 0$.

If $\mu \in (0,1)$, then $|a| \le K$ holds. Otherwise, if $|a| > K$, then by the first part of (H3),

$$\mu\frac{a^2}{\Gamma(\beta+1)} = -(1-\mu)a\Big(I^{\alpha+\beta}f(s,a\frac{s^\beta}{\Gamma(\beta+1)})(1) - \lambda I^{\alpha+\beta+\gamma}f(s,a\frac{s^\beta}{\Gamma(\beta+1)})(\eta)\Big) < 0,$$

which is a contradiction. Hence, $\exists M'$, $\|x\| \le M'$, i.e., the set $\{x \in KerL,\ \mu x + (1-\mu)QNx = 0,\ \mu \in [0,1]\}$ is bounded.

Assume that the other part of (H3) is fulfilled, then we obtain, in a similar way, the boundedness of the set $\{x \in KerL,\ -\mu x + (1-\mu)QNx = 0,\ \mu \in [0,1]\}$. □

To this end, we shall establish the existence theorem for the problem (4.1).

Theorem 4.1. *Let f be continuous and satisfies (H1),(H2) and (H3). The BVP (4.1) possesses a solution on $[0,1]$.*

Proof. Set $\mathscr{U} = \{x \in X, \|x\| < \max(M, M', T) + 1\}$. Then according to Lemmas 4.3 and 4.4, we get

$$Lx \neq \mu Nx, (x,\mu) \in domL \setminus KerL \cap \partial\mathscr{U},$$
$$Nx \notin ImL, x \in KerL \cap \partial\mathscr{U}.$$

Take $H(x,\mu) = \pm\mu x + (1-\mu)QNx$. Employing Lemma 4.5, we obtain $H(x,\mu) \neq 0$ for $x \in KerL \cap \partial\mathscr{U}$. Hence,

$$\begin{aligned} deg(QN\big|_{KerL}, \mathscr{U} \cap KerL, 0) &= deg(H(.,0), \mathscr{U} \cap KerL, 0) \\ &= deg(H(.,1), \mathscr{U} \cap KerL, 0) \\ &= deg(\pm I, \mathscr{U} \cap KerL, 0) \neq 0. \end{aligned}$$

By Theorem 1.9, we infer that the BVP (4.1) possesses a solution on $[0,1]$. □

4.3 Example

To confirm the feasibility and the viability of our outcomes, we present the following example:

$$\begin{aligned} &f(t,x(t)) = e^{-t}(\sin(t) + (t-\xi)|x(t)|), \\ &\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \gamma = \frac{1}{2}, \eta = \frac{1}{2}, \\ &\xi = \frac{1}{2} - (\frac{\pi}{8})^2. \end{aligned}$$

Then

$$\begin{aligned} &|f(t,x(t))| \leq p(t) + q(t)|x(t)|, \\ &p(t) = e^{-t}\sin(t),\ q(t) = |t-\xi|e^{-t}. \\ &R = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+1)} = 2.2732. \end{aligned}$$

Clearly,

$$|q(t)| \leq q(0) = 0.34579.$$
$$\|q\| < \frac{1}{R} = 0.4399.$$

Hence, (H1) is verified. Next, we prove condition (H3).
Let $K = \sqrt{\pi}$ and assume $|a| > K$. We have

$$\frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} = \frac{\Gamma(\beta+\gamma+1)}{\eta^{(\beta+\gamma)}\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma)} = \frac{8}{\pi}.$$

Then

$$\begin{aligned}
&\int_0^1 f(s,a\frac{2\sqrt{s}}{\sqrt{\pi}})ds - \frac{8}{\pi}\int_0^{0.5}(0.5-s)^{0.5} f(s,a\frac{2\sqrt{s}}{\sqrt{\pi}})ds \\
&= \int_0^{0.5}(1-\frac{8}{\pi}(0.5-s)^{0.5})e^{-s}((s-\xi)|a|\frac{2\sqrt{s}}{\sqrt{\pi}}+\sin(s))ds \\
&+ \int_{0.5}^1 e^{-s}((s-\xi)|a|\frac{2\sqrt{s}}{\sqrt{\pi}}+\sin(s))ds > 0.3390.
\end{aligned}$$

Hence,

$$a(\int_0^1 f(s,a\frac{2\sqrt{s}}{\sqrt{\pi}})ds - \frac{8}{\pi}\int_0^{0.5}(0.5-s)^{0.5} f(s,a\frac{2\sqrt{s}}{\sqrt{\pi}})ds) > 0$$

if $a > 0$, and

$$a(\int_0^1 f(s,a\frac{2\sqrt{s}}{\sqrt{\pi}})ds - \frac{8}{\pi}\int_0^{0.5}(0.5-s)^{0.5} f(s,a\frac{2\sqrt{s}}{\sqrt{\pi}})ds) < 0$$

if $a < 0$, hypothesis (H3) holds true. Moreover, the assumption (H2) is verified for any $k > 0$. Indeed,

$$\begin{aligned}
&\int_0^1 f(s,x(s))ds - \frac{8}{\pi}\int_0^{0.5}(0.5-s)^{0.5} f(s,x(s))ds \\
&= \int_0^{0.5}(1-\frac{8}{\pi}(0.5-s)^{0.5})e^{-s}((s-\xi)|x(s)|+\sin(s))ds \\
&+ \int_{0.5}^1 e^{-s}((s-\xi)|x(s)|+\sin(s))ds
\end{aligned}$$

$$> \int_0^{0.5} (1 - \frac{8}{\pi}(0.5 - s)^{0.5}) e^{-s} \sin(s) ds + \int_{0.5}^{1} e^{-s} \sin(s) ds = 0.1562 > 0.$$

Consequently, by Theorem 4.1, the BVP (4.1) admits a solution on $[0,1]$.

Chapter 5

On Multi-term Riemann-Liouville-Caputo Boundary Value Problems

5.1 Introduction

In this strand of research we are concerned with multi-term Riemann-Liouville-Caputo differential equations, combined with fractional natural boundary conditions as follows:

$$\begin{aligned}
&-D^{c}D^{\alpha}u(t)+a^{c}D^{\beta}u(t)+ku(t)=f(t,u(t)), t\in(0,1),\ 0<\alpha<\beta<1,\\
&^{c}D^{\alpha}u(0)=0,\\
&u(1)=\eta^{c}D^{\alpha}u(1).
\end{aligned} \tag{5.1}$$

The mixed derivative $D^{c}D^{\alpha}$ is the Riemannn-Liouville-Caputo (RLC) fractional derivative, which is also called the conservative Caputo derivative. D is the classical derivative. a,k, and η are positive constants. In some particular cases, this equation represents the advection diffusion reaction equation. x represents the concentration of the transported quantity. a is the advection velocity. k is the reaction rate.

The Riemann-Liouville and the Caputo fractional derivatives are frequently studied. There exists a plethora of papers examining the BVPs involving said derivatives and subject to various boundary conditions. However, many applications showed that the novel conservative Caputo derivative, also referred to as the Riemann-Liouville-Caputo (RLC) fractional derivative, lends itself well to certain physical problems; see [74] and the references therein.

As regards the Riemann-Liouville-Caputo fractional derivative, numerical methods are commonly used to inspect their BVPs. In [74], the authors used a finite difference scheme to study the following BVP:

$$
\begin{aligned}
& -D^cD^{\alpha-1}u(t) + b(t)u'(t) + c(t)u(t) = f(t),\, t \in (0,L), \quad 1 < \alpha < 2, \\
& {}^cD^{\alpha-1}u(0) = 0, \\
& u(L) + \beta_1 u'(L) = \gamma_1.
\end{aligned}
$$

For more papers involving the RLC derivative, see [75–79].

Many authors have considered fractional differential equations with a convection term; see [74, 75, 77, 80–91]. In [74–77, 80, 82, 84, 87, 89, 90, 92], the authors have focused on fractional differential equations with a perturbation term. These problems are difficult to handle. Moreover, the inclusion of fractional boundary conditions hinders the problem as it complicates the form of the solution. Nonetheless, recent studies showed that it is possible to overcome the difficulties arising from the appearance of a perturbation term; see [82, 86, 92]. For papers dealing with fractional differential boundary conditions, see [78, 79, 93–99].

Nevertheless, there is a lack of research on fractional multi-term boundary value problems using the RLC fractional derivative with separated Caputo fractional boundary conditions. To our knowledge, no existing paper addresses this specific problem.

The primary focus of this chapter is to undertake a novel situation that is well adapted to applications in the physical context. We conduct our analysis by applying fixed point theory to the transformed Volterra equation. The standard Banach contraction principle as well as the Krasnoselskii's fixed point theorem are applied to prove the existence of solutions.

The present chapter comprises four sections. In Section 2, we give the analytic form of the solution by considering the associated linear problem. Section 3 examines the solvability of the nonlinear problem. We derive sufficient conditions under which the existence of solutions is ensured. Finally, we furnish confirmatory numerical examples in

Section 3.

5.2 Existence results

Using results from fractional calculus, we will prove here an equivalence lemma providing the analytic form of the solution. $C([0,1],\mathbb{R})$ is denoted by $\mathscr{C}$ and equipped with the norm $\|\phi\| = \sup_{t\in[0,1]} |\phi(t)|$. Additionally, we assume that $a \neq \frac{\Gamma(2-\beta)\Gamma(\alpha-\beta+2)}{\Gamma(2-\beta)-\eta\Gamma(\alpha-\beta+2)}$ throughout this chapter.

Lemma 5.1. *Let $h \in \mathscr{C}$. The analogous linear differential problem*

$$\begin{aligned} &-D^c D^\alpha u(t) + a^c D^\beta u(t) + ku(t) = h(t), t \in (0,1),\\ &{}^c D^\alpha(0) = 0,\\ &u(1) = \eta\, {}^c D^\alpha u(1) \end{aligned} \tag{5.2}$$

is equivalent to the BVP

$$\begin{aligned} &-{}^c D^{\alpha+1} u(t) + a^c D^\beta u(t) + ku(t) = h(t), t \in (0,1),\\ &u'(0) = 0,\\ &u(1) = \eta\, {}^c D^\alpha u(1). \end{aligned} \tag{5.3}$$

Moreover, if $\frac{a}{\Gamma(\alpha-\beta+2)} - \frac{\eta a}{\Gamma(2-\beta)} - 1 \neq 0$, then the unique solution of the above problem takes the following form:

$$u(t) = \int_0^1 \mathscr{G}_1(t,s)u(s)ds - \int_0^1 \mathscr{G}_2(t,s)h(s)ds$$

with

$$\mathscr{G}_1(t,s)=\begin{cases} a\Big(\frac{\phi(t)}{\xi}\big(\frac{(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{\eta(1-s)^{-\beta}}{\Gamma(1-\beta)}\big)+\frac{(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\Big)+ & \\ k\Big(\frac{\phi(t)}{\xi}\big(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}-\eta\big)+\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}\Big), & 0\le s\le t\le 1, \\ \frac{a\phi(t)}{\xi}\big(\frac{(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{\eta(1-s)^{-\beta}}{\Gamma(1-\beta)}\big)+\frac{k\phi(t)}{\xi}\big(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}-\eta\big), & 0\le t\le s\le 1, \end{cases} \tag{5.4}$$

and

$$\mathscr{G}_2(t,s)=\begin{cases} \frac{\phi(t)}{\xi}\big(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}-\eta\big)+\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}, & 0\le s\le t\le 1, \\ \frac{\phi(t)}{\xi}\big(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}-\eta\big), & 0\le t\le s\le 1 \end{cases} \tag{5.5}$$

where $\xi=\frac{a\eta}{\Gamma(2-\beta)}-\frac{a}{\Gamma(\alpha-\beta+2)}+1$ *and* $\phi(t)=\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1$. *Moreover,* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *are uniformly bounded, i.e., there exist* $\kappa_1,\kappa_2>0$ *such that*

$$|\int_0^1 \mathscr{G}_1(t,s)ds|\le\kappa_1,\quad |\int_0^1 \mathscr{G}_1(t,s)ds|\le\kappa_2,\ \textit{for every}\, t\in[0,1].$$

Proof. For the first equivalence, see [74]. Next, let u be a solution to (5.3). Apply the operator $I^{\alpha+1}$ to the differential equation

$$-{}^cD^{\alpha+1}u(t)+a\,{}^cD^{\beta}u(t)+ku(t)=h(t),$$

then using (1.3) and (1.4), we obtain

$$-(u(t)+c_0+c_1t)+aI^{\alpha+1-\beta}(u(t)+c_0)+kI^{\alpha+1}u(t)=I^{\alpha+1}h(t).$$

Apply (1.2), then

$$-u(t)-c_0-c_1t+aI^{\alpha-\beta+1}u(t)+\frac{ac_0t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}+kI^{\alpha+1}u(t)=I^{\alpha+1}h(t).$$

Therefore,

$$u(t)=c_0(\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1)-c_1t+aI^{\alpha-\beta+1}u(t)+kI^{\alpha+1}u(t)-I^{\alpha+1}h(t). \tag{5.6}$$

Employing the boundary condition $u'(0)=0$, we obtain $c_1=0$. Now apply the operator ${}^cD^\alpha$ to (5.6); besides using (1.5), (1.6) and (1.7), we retrieve

$${}^cD^\alpha u(t)=\frac{c_0at^{1-\beta}}{\Gamma(2-\beta)}+aI^{1-\beta}u(t)+kI^1u(t)-I^1h(t).$$

The boundary condition $u(1)=\eta^cD^\alpha u(1)$ yields

$$\begin{aligned}c_0(\frac{a}{\Gamma(\alpha-\beta+2)}-1)&+aI^{\alpha-\beta+1}u(1)+kI^{\alpha+1}u(1)-I^{\alpha+1}h(1)\\&=\frac{\eta c_0a}{\Gamma(2-\beta)}+a\eta I^{1-\beta}u(1)+k\eta I^1u(1)-\eta I^1h(1).\end{aligned}$$

Hence,

$$\begin{aligned}&c_0(\frac{a\eta}{\Gamma(2-\beta)}-\frac{a}{\Gamma(\alpha-\beta+2)}+1)\\&\quad=a\Big(I^{\alpha-\beta+1}u(1)-\eta I^{1-\beta}u(1)\Big)+k\Big(I^{\alpha+1}u(1)-\eta I^1u(1)\Big)-\Big(I^{\alpha+1}h(1)-\eta I^1h(1)\Big).\end{aligned}$$

Setting $\xi=\frac{a\eta}{\Gamma(2-\beta)}-\frac{a}{\Gamma(\alpha-\beta+2)}+1$, $\phi(t)=\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1$, and substituting c_0 in (5.6) gives

$$\begin{aligned}u(t)=&\frac{\phi(t)}{\xi}\Big(a(I^{\alpha-\beta+1}u(1)-\eta I^{1-\beta}u(1))+k(I^{\alpha+1}u(1)-\eta I^1u(1))-(I^{\alpha+1}h(1)-\eta I^1h(1))\Big)\\&+aI^{\alpha-\beta+1}u(t)+kI^{\alpha+1}u(t)-I^{\alpha+1}h(t)=\int_0^1\mathscr{G}_1(t,s)u(s)ds-\int_0^1\mathscr{G}_2(t,s)h(s)ds.\end{aligned}$$

Now, we shall prove the uniform boundedness of $\mathscr{G}_1$ and $\mathscr{G}_2$. Indeed,

$$\begin{aligned}|\int_0^1\mathscr{G}_1(t,s)ds|\le a\Big(|\frac{\phi(t)}{\xi}|(\frac{1}{\Gamma(\alpha-\beta+1)}\int_0^1(1-s)^{\alpha-\beta}ds+\frac{\eta}{\Gamma(1-\beta)}\int_0^1(1-s)^{-\beta}ds)\\+\frac{1}{\Gamma(\alpha-\beta+1)}\int_0^t(t-s)^{\alpha-\beta}ds\Big)\end{aligned}$$

$$+k\Big(|\frac{\phi(t)}{\xi}|(\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}ds+\eta)+\frac{1}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}ds\Big)$$
$$\leq a\Big(\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha-\beta+2)}+\frac{\eta}{\Gamma(2-\beta)})+\frac{1}{\Gamma(\alpha-\beta+2)}\Big)$$
$$+k\Big(\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha+2)}+\eta)+\frac{1}{\Gamma(\alpha+2)}\Big):=\kappa_1,$$

where $\nu=\max_{0\leq t\leq 1}|\dfrac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1|=\dfrac{a}{\Gamma(\alpha-\beta+2)}-1$, if $\frac{a}{\Gamma(\alpha-\beta+2)}>1$ and $\nu=1$, if $\frac{a}{\Gamma(\alpha-\beta+2)}\leq 1$. Similarly,

$$|\int_0^1\mathscr{G}_2(t,s)ds|\leq|\frac{\phi(t)}{\xi}|(\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}ds+\eta)+\frac{1}{\Gamma(\alpha+1)}\int_0^1(t-s)^{\alpha}ds$$
$$\leq\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha+2)}+\eta)+\frac{1}{\Gamma(\alpha+2)}:=\kappa_2.$$

□

Now, we present three existence results. The proofs are carried out through fixed point theorems.

Theorem 5.1. *Assume $f\colon[0,1]\times\mathscr{C}\to\mathbb{R}$ is Lipschitzian i.e., there is a constant $L>0$;*

$$|f(t,u)-f(t,v)|\leq L\|u-v\|,\quad t\in[0,b]\ u,v\in\mathscr{C}$$

with $\kappa_1+L\kappa_2<1$. Then the BVP (5.1) admits a unique solution.

Proof. Define the operator $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ as

$$\mathscr{T}u(t)=\int_0^1\mathscr{G}_1(t,s)u(s)ds-\int_0^1\mathscr{G}_2(t,s)f(s,u(s))ds.$$

We shall prove that $\mathscr{T}$ is a contraction mapping. For $u,v\in\mathscr{C}$, $t\in[01]$,

$$|\mathscr{T}u(t)-\mathscr{T}v(t)|\leq\int_0^1|\mathscr{G}_1(t,s)||u(s)-v(s)|ds+\int_0^1|\mathscr{G}_2(t,s)||f(s,u(s))-f(s,v(s))|ds$$
$$\int_0^1|\mathscr{G}_1(t,s)||u(s)-v(s)|ds+L\int_0^1|\mathscr{G}_2(t,s)||u(s)-v(s)|ds$$

$$\leq (\kappa_1 + L\kappa_2)\|u-v\|.$$

Hence, the existence of a unique fixed point follows owing to the Banach contraction principle. □

Theorem 5.2. *If* $\kappa_1 < 1$, *then the BVP (5.1) has at least one solution provided that there exists* $\zeta \in \mathscr{C}$, *verifying* $|f(t,u)| \leq \zeta(t)$, *for every* $u \in \mathscr{C}$, $t \in [0,1]$.

Proof. Let $\mathscr{A}, \mathscr{B} : \mathscr{C} \to \mathscr{C}$ be two operators given by $\mathscr{A}u(t) = \int_0^1 \mathscr{G}_1(t,s)u(s)ds$ and $\mathscr{B}u(t) = -\int_0^1 \mathscr{G}_2(t,s)f(s,u(s))ds$. We easily verify that $\mathscr{A}$ and $\mathscr{B}$ are continuous.
Next, we prove that for every $u,v \in B_r$ we have $\mathscr{A}u + \mathscr{B}v \in B_r$.
$\kappa_1 < 1$ by assumption. Fix a positive number r satisfying $r > \dfrac{\kappa_2\|\zeta\|}{1-\kappa_1}$.
Set $B_r = \{u \in \mathscr{C}, \|u\| \leq r\}$. Hence, for all $u,v \in B_r$,

$$\begin{aligned}
|\mathscr{A}u(t) + \mathscr{B}v(t)| &\leq |\mathscr{A}u(t)| + |\mathscr{B}v(t)| \\
&\leq r\int_0^1 |\mathscr{G}_1(t,s)|ds + \int_0^1 |\mathscr{G}_2(t,s)||f(s,v(s))|ds \\
&\leq r\kappa_1 + \|\zeta\|\kappa_2 < r.
\end{aligned}$$

Now, we shall show that $\mathscr{A}$ is a contraction mapping.

$$\begin{aligned}
|\mathscr{A}u(t) - \mathscr{A}v(t)| &\leq \int_0^1 |\mathscr{G}_1(t,s)||u(s)-v(s)|ds \\
&\leq \|u-v\| \int_0^1 |\mathscr{G}_1(t,s)|ds \leq \kappa_1\|u-v\|.
\end{aligned}$$

Since $\kappa_1 < 1$ by assumption, $\mathscr{A}$ is indeed a contraction mapping.
To this end, we show that $\mathscr{B}$ is compact. Firstly, $\mathscr{B}$ is uniformly bounded on B_r since

$$\begin{aligned}
|\mathscr{B}u(t)| &\leq \int_0^1 |\mathscr{G}_2(t,s)f(s,u(s))|ds \\
&\leq \|\zeta\| \int_0^1 |\mathscr{G}_2(t,s)|ds \leq \|\zeta\|\kappa_2.
\end{aligned}$$

Secondly, we prove that $\mathscr{B}$ is equicontinuous. Let $0 \leq t_1 < t_2 \leq 1$ and $u \in B_r$, then

$$\begin{aligned}
|\mathscr{B}u(t_2) - \mathscr{B}u(t_1)| &= |\int_0^{t_2} \frac{(t_2-s)^\alpha}{\Gamma(\alpha+1)} f(s,u)ds - \frac{1}{\Gamma(\alpha+1)} \int_0^{t_1} \frac{(t_1-s)^\alpha}{\Gamma(\alpha+1)} f(s,u)ds| \\
&+ \frac{|\phi(t_1)-\phi(t_2)|}{\xi} \Big| \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} f(s,u)ds - \eta \int_0^1 f(s,u)ds \Big| \\
&\leq \frac{1}{\Gamma(\alpha+1)} \Big| \int_0^{t_1} \big((t_2-s)^\alpha - (t_1-s)^\alpha\big) f(s,u)ds - \int_{t_1}^{t_2} (t_2-s)^\alpha f(s,u)ds \Big| \\
&+ \frac{a|t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1}|}{|\xi|} \Big(\int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} |f(s,u)|ds + \eta \int_0^1 |f(s,u)|ds \Big) \\
&\leq \frac{\|\zeta\|}{\Gamma(\alpha+1)} \Big(\int_0^{t_1} |(t_2-s)^\alpha - (t_1-s)^\alpha| ds + \int_{t_1}^{t_2} |(t_2-s)^\alpha| ds \Big) \\
&+ \frac{a\|\zeta\|}{|\xi|\Gamma(\alpha-\beta+2)} |t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1}| \big(\frac{1}{\Gamma(\alpha+2)} + \eta \big) \\
&\leq \frac{\|\zeta\|}{\Gamma(\alpha+2)} (t_2^{\alpha+1} - t_1^{\alpha+1}) + \frac{a\|\zeta\|(t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1})}{|\xi|\Gamma(\alpha-\beta+2)} \big(\frac{1}{\Gamma(\alpha+2)} + \eta \big),
\end{aligned}$$

which tends to zero, as t_1 tends to t_2. Employing the Arzelà-Ascoli theorem, we conclude that $\mathscr{B} : \mathscr{C} \to \mathscr{C}$ is completely continuous. Thus, by means of Krasnoselskii's fixed point theorem, $\mathscr{T}$ admits a fixed point $u \in B_r$. Hence, the BVP admits a solution on $[0,1]$. □

Theorem 5.3. *Under the hypotheses:*

1. *f is a continuous function,*
2. *there exist a function $g \in C([0,1],\mathbb{R}_+)$, and a nondecreasing function $\psi \in C(\mathbb{R},\mathbb{R}_+)$ verifying $|f(t,u)| \leq g(t)\psi(\|u\|)$, for every $t \in [0,1]$, $u \in \mathscr{C}$, and*

$$\kappa_1 + \kappa_2 \|g\| \limsup_{r \to +\infty} \frac{\psi(r)}{r} < 1,$$

the BVP (5.1) admits a solution on $[0,1]$.

Proof. Define $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ by

$$\mathscr{T}u(t) = \int_0^1 \mathscr{G}_1(t,s)u(s)ds - \int_0^1 \mathscr{G}_2(t,s)f(s,u(s))ds,$$

as above. In view of the Leray Schauder nonlinear alternative, we shall prove that $\mathscr{T}$ is a continuous and a completely continuous operator.
The proof that $\mathscr{T}$ is continuous is straightforward; hence, it is omitted.
Next, $\mathscr{T}$ is uniformly bounded in $\mathscr{C}$. Let $u \in B_r := \{u \in \mathscr{C}, \|u\| \leq r\}$. Since f satisfies the growth condition,

$$
\begin{aligned}
|\mathscr{T}u(t)| &\leq \int_0^1 |\mathscr{G}_1(t,s)||u(s)|ds + \int_0^1 |\mathscr{G}_2(t,s)||f(s,u(s))|ds \\
&\leq \kappa_1\|u\| + \kappa_2\|g\|\psi(\|u\|) \leq \kappa_1 r + \kappa_2\|g\|\psi(r) := l.
\end{aligned}
$$

Hence, $\|\mathscr{T}u\| \leq l$.
Now, we prove that $\mathscr{T}$ is equicontinuous. Let $0 \leq t_1 < t_2 \leq 1$. Let $u \in B_r := \{u \in \mathscr{C}, \|u\| \leq r\}$. Then

$$
\begin{aligned}
|\mathscr{T}u(t_2) - \mathscr{T}u(t_1)| &\leq |\int_0^1 (\mathscr{G}_1(t_2,s) - \mathscr{G}_1(t_1,s))u(s)ds| + |\int_0^1 (\mathscr{G}_2(t_2,s) - \mathscr{G}_2(t_1,s))f(s,u(s))ds| \\
&:= I_1 + I_2.
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
I_1 =& |\int_0^1 (\mathscr{G}_1(t_2,s) - \mathscr{G}_1(t_1,s))u(s)ds| \\
&\leq |\phi(t_2) - \phi(t_1)|\Big(\frac{a}{|\xi|}\Big|\frac{1}{\Gamma(\alpha-\beta+1)}\int_0^1 (1-s)^{\alpha-\beta}u(s)ds - \\
&\frac{\eta}{\Gamma(1-\beta)}\int_0^1 (1-s)^{-\beta}u(s)ds\Big| + \frac{k}{|\xi|}\Big|\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}ds - \eta\int_0^1 u(s)ds\Big|\Big) \\
&+ \frac{a}{\Gamma(\alpha-\beta+1)}\Big|\int_0^{t_2} (t_2-s)^{\alpha-\beta}u(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-\beta}u(s)ds\Big| \\
&+ k\frac{1}{\Gamma(\alpha+1)}\Big|\int_0^{t_2} (t_2-s)^{\alpha}u(s)ds - \int_0^{t_1} (t_1-s)^{\alpha}u(s)ds\Big| \\
&\leq r\frac{a(t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1})}{\Gamma(\alpha-\beta+2)}\Big(\frac{a}{|\xi|}\big(\frac{1}{\Gamma(\alpha-\beta+2)} + \frac{\eta}{\Gamma(2-\beta)}\big) + \frac{k}{|\xi|}\big(\frac{1}{\Gamma(\alpha+2)} + \eta\big)\Big) \\
&+ r\frac{2a(t_2-t_1)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + r\frac{k(t_2^{\alpha+1} - t_1^{\alpha+1})}{\Gamma(\alpha+2)}.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
I_2 =& \left| \int_0^1 (\mathscr{G}_2(t_2,s) - \mathscr{G}_2(t_1,s)) f(s,u(s)) ds \right| \\
&\le \frac{\|g\|\psi(\|u\|)}{\Gamma(\alpha+1)} \Big(\int_0^{t_1} |(t_2-s)^\alpha - (t_1-s)^\alpha| ds + \int_{t_1}^{t_2} |(t_2-s)^\alpha| ds \Big) \\
&+ \frac{a\|g\|\psi(\|u\|)}{|\xi|\Gamma(\alpha-\beta+2)} |t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1}| \big(\frac{1}{\Gamma(\alpha+2)} + \eta \big) \\
&\le \|g\|\psi(r) \Big(\frac{a(t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1})}{|\xi|\Gamma(\alpha-\beta+2)} \big(\frac{1}{\Gamma(\alpha+2)} + \eta \big) + \frac{1}{\Gamma(\alpha+2)} (t_2^{\alpha+1} - t_1^{\alpha+1}) \Big).
\end{aligned}$$

Both I_1 and I_2 approach zero as t_1 tends to t_2. This finishes the proof of the equicontinuity. Owing to the previous steps along with the Arzelá-Ascoli theorem, $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

A priori estimates

By assumption, we infer the existence of a constant $M > 0$ such that $\kappa_1 M + \kappa_2 \|g\| \psi(M) < M$. Let $\mathscr{U} = \{u \in \mathscr{C}, \|u\| < M\}$. According to the previous steps, $\mathscr{T} : \bar{\mathscr{U}} \to \bar{\mathscr{U}}$ is completely continuous. By contradiction, assume that there exists $u \in \partial\mathscr{U}$, $\lambda \in (0,1)$ satisfying $u = \lambda \mathscr{T} u$. Hence,

$$\begin{aligned}
|u(t)| \le |\lambda \mathscr{T} u(t)| \le |\mathscr{T} u(t)| \le & \int_0^1 |\mathscr{G}_1(t,s)||u(s)| ds + \int_0^1 |\mathscr{G}_2(t,s)||f(s,u(s))| ds \\
&\le \kappa_1 \|u\| + \kappa_2 \|g\| \psi(\|u\|) \le \kappa_1 M + \kappa_2 \|g\| \psi(M) < M,
\end{aligned}$$

which contradicts the fact that $u \in \partial\mathscr{U}$. By virtue of the Leray-Schauder nonlinear alternative, we deduce that $\mathscr{T}$ admits a fixed point in $\bar{\mathscr{U}}$. □

5.3 Examples

Numerical examples are given below to confirm the feasibility and viability of our outcomes.

5.3.1 Example 1

Let us take $\alpha = 0.99$, $\beta = 0.09$, $a = 0.1$, $k = 0.2$, $\eta = 1$, and $f(t,u) = \frac{t+1}{10}\arctan(u)$. Therefore,

$$\Gamma(\alpha-\beta+2) = \Gamma(2.9) = 1.8274,$$
$$\Gamma(2-\beta) = \Gamma(1.91) = 0.9652,$$
$$\Gamma(\alpha+2) = \Gamma(2.99) = 1.9817.$$

Then $\nu = \max\limits_{0\le t\le 1} |\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} - 1| = 1$. Hence,

$$a\Big(\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha-\beta+2)} + \frac{\eta}{\Gamma(2-\beta)}) + \frac{1}{\Gamma(\alpha-\beta+2)}\Big) = 0.2057,$$
$$\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha+2)} + \eta) + \frac{1}{\Gamma(\alpha+2)} = 1.9391,$$
$$\kappa_2 = 1.9391,\ \kappa_1 = 0.2057 + 0.2(1.9391) = 0.5935.$$

Moreover,

$$|f(t,u) - f(t,v)| = \frac{t+1}{10}|\arctan u - \arctan v| \le \frac{t+1}{10}|u-v| \le \frac{2}{10}.$$

Then $\kappa_1 + L\kappa_2 = 0.5935 + 0.3878 = 0.9813$. Consequently, employing Theorem 5.1, the BVP (5.1) possesses a unique solution. Additionally, it is evident that the hypotheses of Theorem 5.2 hold true.

5.3.2 Example 2

Let us take $\alpha = 0.8$, $\beta = 0.5$, $a = 0.1$, $k = 0.2$, $\eta = 0.5$, and $f(t,u) = \frac{t^3}{2}(1 + \frac{\|u\|}{1+\|u\|})$. Clearly $|f(t,u)| \le g(t)\psi(\|u\|)$, where $g(t) = \frac{t^3}{2}$ and $\psi(x) = 1 + \frac{\|x\|}{1+\|x\|}$. Then

$$\Gamma(\alpha-\beta+2) = \Gamma(2.3) = 1.1667,$$

$$\Gamma(2-\beta)=\Gamma(3/2)=\frac{\sqrt{\pi}}{2}=0.8862,$$
$$\Gamma(\alpha+2)=\Gamma(2.8)=1.6765.$$

Then $\nu=\max\limits_{0\le t\le 1}|\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1|=1$. Hence,

$$a\Big(\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha-\beta+2)}+\frac{\eta}{\Gamma(2-\beta)})+\frac{1}{\Gamma(\alpha-\beta+2)}\Big)=0.2321,$$
$$\frac{\nu}{|\xi|}(\frac{1}{\Gamma(\alpha+2)}+\eta)+\frac{1}{\Gamma(\alpha+2)}=1.7261,$$
$$\kappa_2=1.7261,\ \kappa_1=0.2321+0.2(1.7261)=0.5773.$$

Moreover,

$$|f(t,u)-f(t,v)|\le\frac{1}{2}|u-v|.$$

Then $\kappa_1+L\kappa_2=0.5773+0.5(1.7261)=1.0952>1$. We lost the uniqueness of the solution, but it is easy to see that $\kappa_1+\kappa_2\|g\|\limsup\limits_{r\to+\infty}\frac{\psi(r)}{r}<1$, so that Theorem 5.3 is applicable, i.e., the existence of at least one solution is guaranteed.

Conclusions and Perspectives

There are several questions that remain open concerning the equations and results of this thesis. It would be interesting to explore other intricate fractional boundary conditions. Further, it would be desirable to give a much larger class of functions f for which the existence of solutions for each problem can be proved. For the positivity of solutions, it would be of interest to define an ordered Banach space. It would be of considerable interest to give conditions that ensure the stability of the solutions of the given problems.

Within Chapter 2, we have furnished sufficient conditions for the existence, the uniqueness, and the stability of solutions for equations that correspond to some new situations of sequential fractional derivatives associated with infinite delay. Our findings are useful when we study numerical methods for complex systems involving delay. Regarding this work, the significance of our findings resides in the usefulness of our existence results in proving the controllability of a more general system with the aid of semigroup techniques. Notably, when we extend our investigations to fractional differential evolution equations with control, i.e., when the nonlinearity is replaced with $Au(t)+Bv(t)+f(t,u_t)$, where u takes values in a Banach space X, A is a generator of a strongly continuous semigroup of bounded linear operators, B is a bounded linear operator, and the control function v is given in $L^2([0,b],U)$, where U is a Banach space.

Within Chapter 3, we gave various sufficient conditions to ensure the existence of solutions for nonlinear sequential boundary value problems with delay. The utility of our findings resides in the pertinence of the novel problem at hand, which features sequential derivatives and delay differential equations. These notions are relevant to real-world problems. The presented numerical examples enhance both the applicability of the given results and the feasibility of the imposed conditions. For a future expansion of this work, we would like to seek the positivity of solutions employing the associated Green's function and the Guo-Krasnoselskii theorem.

Within Chapter 4, we explore problems that include sequential Caputo fractional derivatives in the differential equation and fractional integrals in the boundary conditions. We validate the practicality of the obtained results through a numerical example, demonstrating how our assumptions are easily verifiable. Continuing along this path, we aim to establish positive solutions for the same problem using the Leggett-Williams theorem. Moreover, another possible augmentation concerns the nonlinearity itself. It would be more general if the nonlinearity depended upon a fractional derivative of the unknown function, since this would enable the consideration of an even broader class of differential equations.

Chapter 5 consists of a fractional BVP where the RLC fractional derivative is featured. We provided three existence uniqueness results by virtue of fixed point theorems. In addition, we established confirmatory examples to enhance our findings. In the same context, future work could explore other fractional derivatives and their impact on multi-term equations. It could also investigate higher-dimensional problems, system stability, and cases on infinite intervals.

These questions will be investigated in further potential future research on this subject.

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